Solving Linear Systems of Equations

$$(*) \quad Ax = b$$

rows columns

$$b \in \mathbb{R}^n, \; x \in \mathbb{R}^m \qquad A \in \mathbb{R}^{n \times m}$$

$(*)$ can be written as

$$\sum_{j=1}^{m} a_{ij} x_j = b_i \qquad i = 1, 2, \ldots, n.$$

(Einstein notation) $a_{ij} x_j = b_i$, the i^{th} row.

A is sometimes called the coefficient matrix.

$(A|b)$ is the "augmented matrix"

METHOD OF SOLUTION: Most general method is Gauss-Jordan (or row reduction) strategy.

Take a numerical linear algebra to learn how one solves commonly-occurring high dimensional problems efficiently on modern computers.

The basic idea is to find an equivalent system that can be solved via backsubstitution.

Equivalent Systems have matrices A & B for which $Ax=b$ and $Bx=c$ have the same x solution.

Elementary Row Operations:

1. Row exchanges
2. Row Scaling
3. Row Replacement

The can all be expressed as (left) matrix multiplications.

Let E_i represent the i^{th} row of the augmented matrix.

Row Exchange $\quad j \leftrightarrows k$ rows

$$E_j \rightleftarrows E_k$$

Scaling the j^{th} row: αE_j
(α a scalar)

$$E_j \to \alpha E_j$$

Row Replacement

$$E_j \longrightarrow \alpha E_i + E_j$$

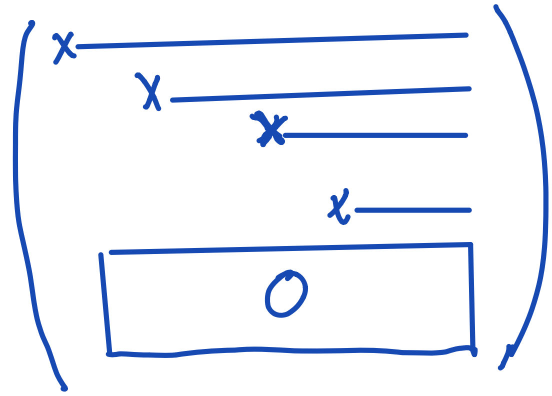
Echelon Form

(1) All zero rows (if any) are below non-zero rows.

(2) Non-zero row: first leading entry (left side) is strictly to the right of any previous row.

(3) The leading entry can be 1 or 0.

Row Echelon Structure



(*) $Ax = b$

(*) is inconsistent (no solution) iff $\exists$ a row of the equivalent augmented matrix system of the form

$$\left(\underbrace{0 \; 0 \cdots\cdots 0}_{} \,\middle|\, \alpha \right) \qquad \alpha \neq 0$$

i.e. $0x_1 + 0x_2 \ldots 0x_m = \alpha \qquad \alpha \neq 0$

Consistent and infinite # of solutions:

there are <u>free</u> variables.

Consistent and unique solution: the equivalent B has a pivot in every row and column.

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Solutions and Systems and their connection to notions of linear independence

S'pose $\{v_i\}_{i=1}^{m}$ vectors and each $v_i \in \mathbb{F}^n$, form

$$A \equiv \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_m \end{pmatrix} \in \mathbb{F}^{n \times m}$$

(i) v_i are LI iff A has a pivot in every column.

(ii) $\{v_i\}$ is "spanning" iff A has a pivot in every row.

(iii) $\{v_i\}_{i=1}^{m}$ are a basis for $\mathbb{F}^n$ iff A has a pivot in every row & column.

Pf: Form $\quad \emptyset = x_1 v_1 + x_2 v_2 \cdots + x_m v_m$

or $\quad \emptyset = A x$

the v_i's are LI iff $x_1 = x_2 = \cdots x_m = 0$

i.e. iff $Ax = \phi$ has only $\underset{\sim}{x} = 0$ as a solution.

Can form $Ax = 0$ and use row ops to test LI of vectors $\{v_i\}_{i=1}^{m}$

$$A = \begin{pmatrix} | & | & | & | & | \\ v_1 & v_2 & \cdots & & v_m \\ | & | & | & | & | \end{pmatrix}$$

Can happen iff (i) is true.

Now, consider $Ax = b$, $b \neq \phi$. Interpret as $b = Ax$ (b must be in the span of $\{v_i\}$, the columns of A) for this problem to have a solution.

can only happen if (ii) holds.

Ask now: when does $Ax = b$ have a <u>unique</u> solution?

iff every row/column has 1 pivot.

can only happen if (iii) holds.

Th'm: An LI set of vectors in $\mathbb{F}^n$ cannot have more than n vectors (it can have less).

Pf: Take $v_1, \dots, v_m \in \mathbb{F}^n$.

form $A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}$.

We know that $Ax = 0$ has only $x = 0$ as a solution iff every row & column has 1 pivot $\Rightarrow m \not> n$ //

Rmk: Any basis for $\mathbb{F}^n$ must have exactly n vectors

$b = Ax \qquad b \in \mathbb{F}^n$ //

Invertible Matrices: A is invertible

iff A has pivots in every row & column.

Hence A must be square.

If $A \in \mathbb{F}^{n \times n}$ & is left or right invertible

$\Rightarrow$ it is invertible: hence,

need to check that $A^{-1}A = I$

or $AA^{-1} = I$ to conclude that A^{-1} exists.

If A is invertible $\Rightarrow$

$Ax = b$ has solution $x = A^{-1}b$

Find A^{-1} by row reduction

Form: $(A|I) \xrightarrow{\text{use row ops to find}} (I|B)$

$\Rightarrow B = A^{-1}$. What is going on?

$(A|I)$ is really $Ax_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$, $Ax_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \end{pmatrix}$, ... $Ax_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$.

$Ax_k = \hat{e}_k \quad k=1,2,\dots,n$ simultaneously

$$\hat{e}_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow k^{\text{th}} \text{ entry.}$$

$$A \underbrace{\begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix}}_{B} = I \qquad B = A^{-1}$$

Dimension (finite dim) spaces

$\dim(V)$ = # of vectors in a basis for V

e.g. $\dim(\phi) = 0$

$\dim(\mathbb{F}^{\infty}) = \infty$

A vector space is finite-dimensional iff it has a finite spanning space.

If $\dim(V) = n$ $\exists$ an isomorphism $A : V \to \mathbb{R}^n$. In fact a basis $v_1, v_2, \ldots, v_n \in V$ can be constructed:

$$\hat{e}_k = A v_k,$$ where

$\{\hat{e}_k\}_{k=1}^{n}$ are the basis of $\mathbb{R}^n$

General Solution of $Ax = b$

The homogeneous problem is defined as

$$(*) \quad Ax = \phi,$$

it has a solution x_h.

x_h can be ϕ. In fact, the trivial solution ϕ is always a solution of $(*)$

The nonhomogeneous solution x_p is the solution of $Ax_p = b$. Hence, the general solution to $Ax = b$ is

$x = x_h + x_p$. Note that

$$\boxed{A x = \underset{0}{A x_h} + A x_p = b}$$

ex) Consider the differential equation

$$\mathcal{L} y = f(x), \text{ where } \begin{cases} y = y(x) \in \mathbb{R}^1 \\ f = f(x) \in \mathbb{R}^1 \end{cases}$$

$\mathcal{L}$ is a linear differential operator

$$\mathcal{L} = a^n \frac{d^n}{dx^n} + a^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a^0, \quad a^i \in \mathbb{R}^1$$

$\mathcal{L}$ is a linear transformation $\mathcal{L} : V \to V$, $V = \mathbb{R}^1$.

The solution is $y = y_h + y_p$,

where $\mathcal{L} y_h = 0$ and $\mathcal{L} y_p = f(x)$.