LADR Sections 3A, 3C, 3D

Properties of Matrix Multiplication

Associativity $A(BC) = (AB)C$

Distributivity $A(B+C) = AB + AC$

Scalar Multiplication $A(\alpha B) = \alpha AB$, α a scalar.

(the products involving A, B, C have to make sense dimensionally)

Commutivity? $AB \neq BA$ generally even if both are $n \times n$

def: The Commutator for 2 operators (or matrices) A and B $\quad C \equiv [A,B] = AB - BA$

If $[A,B] = 0 \Rightarrow A$ and B commute.

The Transpose of a Matrix (A^T)

If $A \in \mathbb{R}^{n \times m}$ consisting of $A = (a_{ij})$

Then $A^T = (a_{ji})$.

ex) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$ //

$$\boxed{(AB)^T = B^T A^T}$$

Trace of a Matrix: $Tr(A) = \sum_{i=1}^{n} a_{ii}$

for $A \in \mathbb{F}^{n \times n}$

$$\boxed{Tr(AB) = Tr(BA)}$$

//

Invertible Transformations

The Identity Matrix: $I_n = diag(1)_n$

$$I_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Sometimes we write I and the dimension of I

is understood from the context.

if $x \in \mathbb{R}^n$ $\quad I_n x = x$, $I_n \in \mathbb{R}^{n\times n}$.

if $A \in \mathbb{R}^{n\times n}$ $\quad I_n A = A$

//

More generally, if V is a vector space and $x \in V$

$\exists$ I s.t. $Ix = x$

$$T(x) = I(x) : V \to V$$

//

Invertible Transformation (1) Let $A: V \to W$ be a linear transformation. We say that A is left invertible if $\exists$ a linear transformation $B: W \to V$ st

$$BA = I \in V$$

(2) The transformation is right invertible if $\exists$ a linear transformation $C: W \to V$ st

$$AC = I \in W$$

B & C are the left/right inverses of A.

No uniqueness is implied here.

(3) A linear transformation is invertible if it is BOTH left and right invertible

i.e $AD = I = DA$

$[A,D] = 0$

<u>Th'm</u>: The inverse of a linear map is unique if it is invertible

<u>Pf</u>: Let $T: V \to W$ for which an inverse exists.

Let S_1 and S_2 be inverses of T.

then $S_1 = S_1 I = S_1(TS_2) = (S_1 T)S_2 = IS_2 = S_2$

$\therefore \; S_1 = S_2$ //

Some simple examples of inverses:

ex) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has a left inverse, but not right inverse:

$\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/3 & 2/3 \end{pmatrix}$ are 2 such left inverses.

ex) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has a right inverse, but not left inverse.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a right inverse, for example.

ex) $\alpha \in \mathbb{R}^1$ has a right and left inverse

$\frac{1}{\alpha}\alpha = 1$ and $\alpha \frac{1}{\alpha} = 1$

How do we handle $\alpha = 0$ case?

* The invertibility of A & B does not imply that (AB) is invertible.
* The invertibility of (AB) and A (or B) implies the invertibility of B (or A)

Th'm (Inverse of A^T): If A is invertible $\Rightarrow$ A^T is invertible:

$$(A^T)^{-1} = (A^{-1})^T$$

Pf: If A is invertible, so is A^{-1}. If A & B are invertible and AB is defined

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

//

Isomorphic Spaces:

If 2 vector spaces have the same properties and equivalent constructs, e.g. preserved operations, we call these 2 spaces isomorphic.

By using the basis sets one can elegantly show that 2 spaces are isomorphic:

Th'm: Let $A: V \to W$ be an isomorphism. Let $v_1, v_2, \ldots, v_p$ be a basis of V. Then the set $Av_1, Av_2, \dots, Av_p$ is a basis for W. //

* An isomorphism is an invertible linear map!
* 2 finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic iff they have the same dimension.

Invertibility and sets of linear equations:

Thm: Let $A: X \to Y$ be a linear transformation. A is invertible iff, for $b \in Y$

$$Ax = b$$

has a solution $x \in X$, unique. //