LINEAR TRANSFORMATIONS

Read 3A&3C LADR

Let X, Y be 2 vector spaces,

def: $y = T(x)$ denotes a transformation.

☆ $y = T(x)$ is a linear transformation.
Input $x \in X$, the output is $y \in Y$
$X, Y \in V$ (denoting vector spaces)
So T assigns a "value" y to every x.

Another way to write this

$$T : X \to Y$$

domain (X), range (Y) (target) (co-domain)

What makes this a linear transformation?

by ☆ it maps a vector space to a vector space.

BUT WHAT MAKES IT A LINEAR TRANSFORMATION?

def: A linear transformation: Let V, W be vector spaces. The transformation $T: V \to W$ is called linear if

(a) $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in V$

(b) $T(\alpha v) = \alpha T(v) \quad \forall\, v \in V \;\&\; \alpha \in \mathbb{F}^1$

(a) is "linear superposition principle".

Can combine (a) & (b) as

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

$$\forall\, u, v \in V \text{ and } \alpha, \beta \in \mathbb{F}^1$$

It helps to see what is NOT a linear transformation:

ex) A very simple nonlinear transformation:
let $x \in \mathbb{R}$.

$T: x \to x^2 \quad$ or $\quad y = x^2$

$T(\alpha x) = \alpha^2 x^2$

$T(\alpha x + \beta x) = (\alpha x + \beta x)^2 = \alpha^2 x^2 + 2\alpha\beta x^2 + \beta^2 x^2$

$\neq T(\alpha x) + T(\beta x)$

ex) $T: f(x) \longrightarrow \frac{df}{dx}$. Assume $f(x) \in C^{\infty}(\mathbb{R})$.

is this a linear transformation?

[$f \in C^{\infty}(\mathbb{R})$ are functions that are continuous and have continuous derivatives at all orders. e.g., $f = \cos x$, e^x, etc, for all $\mathbb{R}$.

Let $f, g \in C^{\infty}(\mathbb{R})$. Note $\frac{df}{dx} \in C^{\infty}(\mathbb{R})$, $\frac{dg}{dx} \in C^{\infty}(\mathbb{R})$.

A $T(f) = \frac{df}{dx}$ $\quad \alpha T(f) = \alpha \frac{df}{dx}$ $\quad \alpha, \beta \in \mathbb{F}$

$T(g) = \frac{dg}{dx}$ $\quad \beta T(g) = \beta \frac{dg}{dx}$

B $T(\alpha f + \beta g) = \frac{d}{dx}(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ //

Linear Maps take $\emptyset \in V$ to $\emptyset \in W$

Suppose T is a linear transformation from $V \to W$

$\Rightarrow T(\emptyset) = \emptyset$.

Pf: $T(\emptyset) = T(\emptyset + \emptyset) = T(\emptyset) + T(\emptyset)$

$\alpha T(\emptyset) = T(\alpha \emptyset)$

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Basic linear transformations on $\mathbb{R}^2$

1. DILATION
2. REFLECTION
3. ROTATION
4. PROJECTION
5. SHEARING

All of these are linear transformations that map V into itself: $T: V \to V$ $\quad V \in \mathbb{R}^2$.

Let $u, v \in V$

then $v = Tu \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a, b, c, d \in \mathbb{R}^1$

1. Dilation: $v = \lambda u$ (uniform) $\qquad \lambda > 1$ dilation

$0 < \lambda < 1$ contraction

$T = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Tu = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

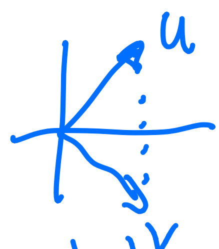
$u = \begin{pmatrix} x \\ y \end{pmatrix} \qquad = \lambda u$

Non-uniform dilation:

$$T = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad v = Tu = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix} \quad //$$

(2) Reflection: $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

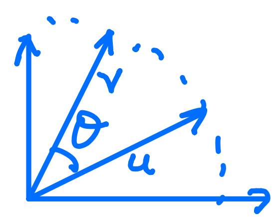
$$v = Tu = \begin{pmatrix} x \\ -y \end{pmatrix}$$



reflection about x axis

$T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection about y axis //

(3) Rotation: Rotation by an angle θ:



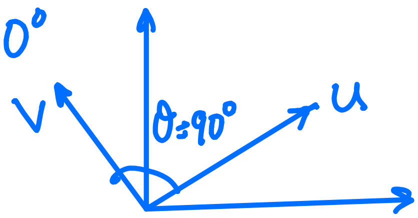
$$v = \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad u = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Show that T is the linear transformation.

eg) Rotate by 90°

$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

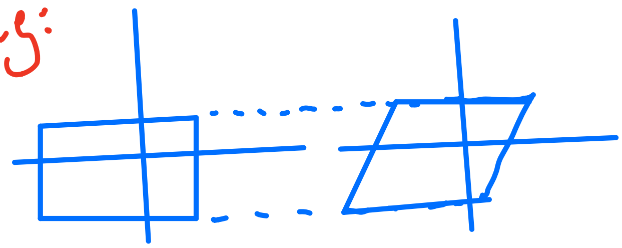


④ Projection:

$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad u = \begin{pmatrix} x \\ y \end{pmatrix}$

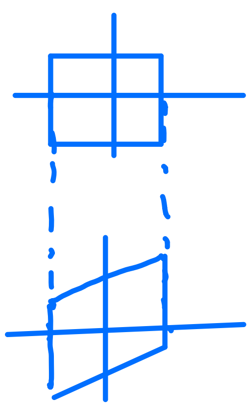
$v = \begin{pmatrix} x \\ 0 \end{pmatrix}$

⑤ Shearing:



$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad u = \begin{pmatrix} x \\ y \end{pmatrix}$

$Tu = \begin{pmatrix} x+y \\ y \end{pmatrix}$



$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$v = \begin{pmatrix} x \\ x+y \end{pmatrix}$

LINEAR TRANSFORMATIONS AS A VECTOR SPACE

1. Multiplication by a scalar:

Let $T: V \to W$, $v_i \in V$ $i = 1, 2, 3$; $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}^1$; V, W are vector spaces

$$(\alpha T)v = T(\alpha v) = \alpha T v$$

$$(\alpha_1 T)(\alpha_2 v_1 + \alpha_3 v_2) = \alpha_1 \alpha_2 T v_1 + \alpha_1 \alpha_3 T v_2$$
$$= \alpha_1 T(\alpha_2 v_1) + \alpha_1 T(\alpha_3 v_2)$$

2. If T_1 & T_2 have same domain & range:

Let $T_1: V \to W$ & $T_2: V \to W$

then $T = T_1 + T_2$, $v \in V$

is $Tv = T_1 v + T_2 v \in W$. //

If $T: \mathbb{R}^n \to \mathbb{R}^m$

T is a rectangular matrix:

$$T \in \mathbb{R}^{\overbrace{m}^{\text{rows}} \times \overbrace{n}^{\text{column}}}$$

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Composition of linear transformations & matrix multiplication

def: Multiplication by Matrix: We generalize matrix-vector multiplication: AB is generated by multiplying each column of B by the matrix A

If $\underline{b}_1, \underline{b}_2 \dots \underline{b}_r$ are columns of $B \Rightarrow$

$A\underline{b}_1, A\underline{b}_2, \dots, A\underline{b}_r$ are the columns of AB

The entry $(AB)_{jk}$ (j^{th} row, k column)

$= (\text{row of } A) \times (\text{column } k \text{ of } B)$

$$(AB)_{jk} = \sum_{\ell} a_{j\ell} b_{\ell k} \equiv a_{j\ell} b_{\ell k}$$

(Einstein Notation: repeated indices imply sums) ex) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$ what is

$C = AB$?

$A \in \mathbb{F}^{2\times 3}$ $\qquad$ $B \in \mathbb{F}^{3\times 2}$

$C = AB \in \mathbb{F}^{2\times 2}$ $\qquad$ B

A $\qquad$ $AB \in \mathbb{F}^{2\times 2}$

so $C = AB$, $c_{jk} = \sum_{\ell=1}^{3} a_{j\ell} b_{\ell k} = c_{jk}$

$j,k = 1,2$ //

$C = AB$ is only defined if

$A \in \mathbb{F}^{m\times n}$ $\quad$ $B \in \mathbb{F}^{n\times r}$

and $\quad C = \mathbb{F}^{m\times r}$.