# LADR 1C & Chapter 2

A set of $p$ vectors $\{u_i\}_{i=1}^{p}$ is said to be LINEARLY INDEPENDENT (LI) iff the only value ALL constants $\{\alpha_i\}$ $i=1,2..,p$ that makes

$$\alpha_1 u_1 + \alpha_2 u_2 \dots + \alpha_p u_p = 0$$

are $\alpha_1 = \alpha_2 \dots = \alpha_p = 0.$

Otherwise, we say that the (whole) set is LINEARLY DEPENDENT.

**Linear Dependence Lemma**: S'pse $v_i$ $i=1,\dots,p$. are linearly dependent (LD) in $V$. $\Rightarrow \exists$ at least one $j \in \{1,2..,p\}$ s.t. the following holds:

$$v_j \in \operatorname{span}(v_1, v_2 \dots v_{j-1}, v_{j+1}, \dots, v_p).$$

(✱) If the $j$th term is removed from the set $v_1, v_2 \dots, v_p$ the span of the remaining set equals $\operatorname{span}(v_1, v_2 \dots, v_p)$

Pf: since vectors are LD $\exists\ \alpha_1, \ldots \alpha_p \in \mathbb{F}$, not all 0, st

$$\alpha_1 v_1 \cdots + \alpha_p v_p = 0.$$

Let $j$ be the largest of $\{1, 2, \ldots, p\}$ st $\alpha_j \neq 0$.

Then

$$(*)\quad v_j = -\frac{\alpha_1}{\alpha_j} v_1 - \frac{\alpha_2}{\alpha_j} v_2 \cdots - \frac{\alpha_{j-1}}{\alpha_j} v_{j-1},$$

S'pse $u \in \operatorname{span}(v_1, v_2, \ldots v_p) \Rightarrow \exists$ $c_1, c_2 \ldots c_p \in \mathbb{F}$ st

$$(**)\quad u = c_1 v_1 + c_2 v_2 \cdots c_p v_p.$$

Replace $v_j$ in $(**)$ with $(*)$. Easy to see that $u$ is in the span of the set obtained by removing the $j^{th}$ term of $v_1, \ldots v_p$ $\therefore$ $(\#)$ holds //

# Subspaces

def: A subset of $V$, called $U$ is a **subspace** of $V$ if $U$ itself is a vector space.

A subspace must be a vector space $\therefore$ it must be closed under addition, multiplication, must have a unique inverse, must have $\emptyset$, etc.

ex) $\mathbb{R}^2$, $\{0\}$, and all lines passing through the origin are subspaces of $\mathbb{R}^2$ //

ex) $\mathbb{R}^2$, planes passing through origin, $\{0\}$, $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ //

Consider the following question:

Is the union of all subspaces a subspace? No.

The union of ALL subspaces is a subspace iff these subspaces are contained in each other //

# SUMS OF SETS AND OF SUBSPACES

Spse $U_1, U_2, \ldots, U_m$ are subsets of $U$. The sum of $U_1, U_2, \ldots, U_m$ is denoted $U_1 + U_2 + \cdots + U_m$. This is to be interpreted to say

$$U_1 + U_2 \cdots + U_m \equiv \{u_1 + \cdots + u_m : u_i \in U_i \quad i = 1, \ldots, m\}$$

ex) let $U = \{(x,0,0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$

$V = \{(0,y,0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$

then $U + V = \{(x,y,0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$. //

In fact in the above example the subsets happen to be <u>subspaces</u> //

<u>The Direct Sum</u>: Suppose $\{U_i\}_{i=1}^{m}$ are subspaces of $U$

The direct sum: $U_1 \oplus U_2 \oplus \cdots \oplus U_m$

indicates $u_1 + u_2 \cdots + u_m$ where each $u_i \in U_i$ $i = 1 \ldots m$.

ex) Let $U = \{(x, y, 0) \mid x, y \in \mathbb{F}\}$, $V = \{(0,0,z) \mid z \in \mathbb{F}\}$

$\mathbb{F}^3 = U \oplus V$

Spse $U$ & $V$ are subspaces of $W$. Then

$U + V$ is a DIRECT SUM iff $V \cap U = \{\emptyset\}$

Condition for a direct sum: $\{U_i\}_{i=1}^{M}$ subspaces of $U$, their sum

$U_1 + U_2 \ldots U_M$ is a direct sum

iff the only way to write $0$ as a sum $u_1 + u_2 \cdots u_m$ is if $u_i = 0$ $i = 1, \ldots, m$.

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# BASIS, Revisited

def: A basis of $V$ is a list of vectors in $V$ that is LI and spans $V$.

ex) $(1,2), (3,5)$ form a basis for $\mathbb{R}^2$. But the set $(1,2), (3,5), (4,13)$ span $\mathbb{R}^2$ but is not a basis for $\mathbb{R}^2$

Q: Why?

Every spanning set of $V$ contains a basis of $V$ $\Rightarrow$ one can always find the basis of $V$ knowing its span, at least formally:
Spse $\{v_i\}_{i=1}^{m}$ span $V$. To find basis we systematically remove vectors with the ultimate aim of exposing the basis set:

1. Let $B = \{v_i\}_{i=1}^{m}$
2. If $v_1 = 0$ delete $v_1$ $\Rightarrow$ leaves $B$ unchanged.
3. for $j = 2, \dots m$
   If $v_j$ is in span $\{v_1 \dots v_{j-1}\}$, delete $v_j$
   otherwise leave $B$ unchanged.
   end

$B = \{v_i\}_{i=1}^{n}$, $n \leq m$

**Lemma:** Every finite-dimensional vector space has a basis.

**Lemma:** Every subspace of $V$ is part of a $\oplus$ equal to $V = U \oplus W \oplus Z \oplus \dots$

**def:** Dimension of a vector space: $\dim(V)$ is the length of the basis set of $V$

eg) $\mathbb{R}^n$ has $\dim(\mathbb{R}^n) = n$ //

ex) $\dim \mathcal{P}^m(\mathbb{R}) = m+1$

$$\mathcal{P}^m(\mathbb{R}) = \{1, x, x^2, \ldots, x^m\}$$ //

$\dim(U) \leq \dim(V)$ if $V$ is finite dimensional and $U$ is subspace of $V$ //

**Lemma:** In a vector space $V$, the span of any subset is a subspace.

**Pf:** (i) If $V$ is empty $\Rightarrow \text{span}(V) =$ trivial subspace.

(ii) If $V$ is not empty, need to check that the span$[S]$ is closed under linear combinations of pairs of elements of the subset $S$:

Take $\underset{\sim}{v} = c_1 s_1 + c_2 s_2 \cdots c_n s_n$,

$\underline{w} = c_{n+1} s_{n+1} + c_{n+2} s_{n+2} \cdots c_m s_m$

$p(c_1 s_1 + c_2 s_2 \cdots c_n s_n) + r(c_{n+1} s_{n+1} \cdots c_m s_m)$

$= pc_1 s_1 + pc_2 s_2 \cdots pc_n s_n + rc_{n+1} s_{n+1} \cdots$
$+ rc_m s_m$

is a linear combination of elements of $V$, and itself an element of $V$, since $S \in V$.

**Note:** possibly some of the $s_i$'s of $\underset{\sim}{v}$ are equal to the $s_j$'s of $\underset{\sim}{w}$ but that does not matter.