SVD

If $A \in \mathbb{F}^{m\times n}$ then

$$A = U_1 \Sigma U_2^*$$

if $A \in \mathbb{R}^{m\times n}$

$$A = Q_1 \Sigma Q_2^*$$

if A is symmetric positive definite

$$A = Q \Sigma Q^T$$

Where Σ diagonals are (the nonzero) square root of e'values of AA^T (or A^TA) //

Rmk: The cols of Q_1 & Q_2 are orthogonal bases for the 4 fundamental subspaces of A:

first r cols of Q_1 $= \mathrm{col}(A)$

last $m-r$ cols of Q_1 $= \mathrm{null}(A^T)$

first r cols of Q_2 = row(A)
last n-r cols of Q_2 = null(A)

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Rmk: The SVD has special structure:
If A multiplies a column of Q_2 it produces a multiple of a column of Q_1:

$$A = Q_1 \Sigma Q_2^*$$

$$AQ_2 = Q_1 \Sigma Q_2^* Q_2 = Q_1 \Sigma$$

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Rmk: The connection with AA^T & A^TA and the SVD $A = Q_1 \Sigma Q_2^*$ follows from:

$$AA^T = (Q_1 \Sigma Q_2^*)(Q_1 \Sigma Q_2^*)^T = Q_1 \Sigma \Sigma^T Q_1^*$$

$$A^TA = Q_2 \Sigma^T \Sigma Q_2^*$$

(A) $AA^TQ_1 = Q_1 \underbrace{\Sigma\Sigma^T}_{m \times m}$ (e'vcle problem, $m \times m$)

(B) $A^TAQ_2 = Q_2 \underbrace{\Sigma^T\Sigma}_{n \times n}$ (e'vclue, $n \times n$)

both have e'vclues $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ on the diagonals.

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ex) Suppose A is diagonal $A \in \mathbb{R}^{\underset{m}{3}\times\underset{n}{2}}$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}, \text{ then } \quad A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \quad AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = Q_1 \Sigma Q_2^T = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q_2^T}$$

Use (A) & (B) to find these. //

ex) Let $A \in \mathbb{R}^{3\times 1}$, for example

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \text{ then } \begin{cases} A^TA = 9 \\ AA^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \end{cases}$$

$$A = Q_1 \Sigma Q_2^T = \underbrace{\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{Q_2^T}$$

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SVD and LSQ: The pseudo inverse A^+ and the minimal norm, LSQ estimate:

Recall LSQ : $A^TA\hat{x} = A^Tb$ (the Normal Eq's)

The columns of A must be LI, i.e. $\text{rank}(A) = n$!

otherwise $(A^TA)^{-1}$ doesn't exist.

so $\hat{x} = (A^TA)^{-1}A^Tb$ can't be computed!

If $\text{rank}(A) < n$ then any x_n in the null space ($Ax_n = 0$) can be added: $\hat{x} + x_n$. What to pick for x_n?

Two possible difficulties with find $Ax = b$

(1) Rows of A are dependent (may not have a solution)

We then settle for projecting b onto $\text{col}(A)$

and solve $A\hat{x} = Pb$

(2) $\text{Col}(A)$ are dependent

Solution may not be unique, i.e.

Can still get $A\hat{x} = Pb$

Let's address case (2):

The question is: What to pick for $\hat{x}$?

Why not choose the $\hat{x}$ with smallest length. Call that x^+.

To Find x^+, we introduce the

Pseudo-inverse A^+

We will find these via SVD.