We'll test our command of the concepts introduced thus far with a variety of examples:

ex) Is the set of 2-tuples $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in V$ where $n_i \in \mathbb{Z}$, a vector space?

not closed under multiplication:

for example, pick $\alpha = \pi$. Form $u = \pi \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$.

Note that $u \notin V$. //

ex) Name the zero vector

(a) Space of 2x4 matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) The space $\{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$

$f(x) = 0 \quad x \in [0,1]$ //

Find the additive inverse of $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$:

Find u s.t. $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $u = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$ //

Is $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x+y+z=1 \right\}$ a vector space?

$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad a+b+c=1$

$v = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ $\quad \alpha+\beta+\gamma = 1$?

Not closed under multiplication: pick $a=1$
$b=c=0 \Rightarrow v \notin U$

Also not closed under addition

pick $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = u$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v$ are in U but $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = u+v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin U$ //

Is $V = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}^+\}$ a vector space?

Not closed under multiplication, e.g.

$-1(a_0 + a_1 x + a_2 x^2) \notin V$ //

Q: Is the set of all rationals a vector space?

No... why??

//

Q: Is the set of all matrices a linear vector space?

No. Take any 2x2 and a 3x3:

Not closed under multiplication, or addition

//

Is the set of polynomials of degree ≥ 2, along the 0 polynomial, a vector space?

Let V be such a set.

let $u = x^2 \in V$

$v = 1 + x - x^2 \in V$

$u + v = 1 + x \notin V$

//

LINEAR COMBINATIONS, LINEAR INDEPENDENCE

BASIS
SPAN
LINEAR COMBINATION

$\left.\begin{matrix}\text{BASIS}\\ \text{SPAN}\\ \text{LINEAR COMBINATION}\end{matrix}\right\}$ definitions to master

DEF: Linear Combination: $\underset{\sim}{w} = \sum_{i=1}^{p} \alpha_i \underset{\sim}{v_i}$

$\alpha_i \in \mathbb{F}^1$

$w = \alpha_1 \underline{v_1} + \alpha_2 \underline{v_2} \cdots \alpha_p \underline{v_p}$

α_i are called "coordinates"

here $\underline{v_i} \in V$

$\Rightarrow \underset{\sim}{w} \in V$

Def: The Basis of V, a set $\{\underset{\sim}{v_i}\}_{i=1}^{p}$

is a basis of V if for $\underset{\sim}{w} \in V$

$\underset{\sim}{w} = \sum_{i=1}^{p} \alpha_i \underset{\sim}{v_i}$

this linear combination is unique.

A set of vectors $\underset{\sim}{w}_1, \underset{\sim}{w}_2 \ldots \underset{\sim}{w}_p \in V$ "span" V if any vector $\underset{\sim}{w} \in V$ admits $\underset{\sim}{w} = \sum_{i=1}^{p} \beta_i \underset{\sim}{w}_i$

A basis of V ALSO SPANS V.

The basis set $\{\underset{\sim}{v}_i\}_{i=1}^{p}$ is a unique spanning set of V

$$\underset{\sim}{w} = \sum_{i=1}^{p} \alpha_i v_i \quad , \text{i.e.}$$

the α_i are uniquely defined.

Def: A trivial combination is $\alpha_k = 0$ $k=1,2,\ldots,p$, in $\sum_{i=1}^{p} \alpha_i \underset{\sim}{v}_i = \emptyset$

Suppose $v_i \in V$ $i=1,\ldots p$ and $w \in \text{span}(v_1, v_2, \ldots, v_p)$

$\exists \{a_i\}_{i=1}^{p}$ st $\underset{\sim}{w} = \sum_{i=1}^{p} a_i \underset{\sim}{v}_i$

and suppose $\{c_i\}_{i=1}^{p}$ is another set of coordinates s.t.

$$\underline{w} = \sum_{i=1}^{p} c_i \underline{v}_i$$

$$\Rightarrow \emptyset = \sum_{i=1}^{p} (a_i - c_i) \underline{v}_i$$

If the only way to do this is if $\alpha_i = (a_i - c_i)$ forms a trivial combination

$$\emptyset = \sum_{i=1}^{p} \alpha_i \underline{v}_i \quad , \text{i.e. } a_i = c_i$$

the linear combination is unique.

$\Rightarrow \{v_i\}_{i=1}^{p}$ forms a basis for V

(exists, unique)

//

Pick an example that highlights uniqueness:

ex) $\mathbb{R}^2$ $\{(0,1), (1,0), (1,1)\}$ span ex) Let $V = \mathbb{R}^2$

The set $\{(0,1), (1,0), (1,1)\}$ span V.

Note that $(1,1) = (0,1) + (1,0)$

So $(1,1)$ is a linear combination of $(0,1)$ & $(1,0)$.

Note any $(a,b) \in V$, $a,b \in \mathbb{R}^1$

$(a,b) = a(1,0) + b(0,1)$

So any $(a,b) \in V$ can be written as a (unique) linear combination of $(1,0)$ and $(0,1)$

A basis for $\mathbb{R}^2$ is $\{(0,1), (1,0)\}$ //

def: (LI) Linearly independent: A list of $v_i \in V$ is LI iff the only choice of $\alpha_i \in \mathbb{F}$ that makes

$$\emptyset = \sum_{i=1}^{p} \alpha_i \underset{\sim}{v_i} \text{ is } \alpha_i = 0 \quad i=1,2..p.$$

//

If not LI then we say the v_i's are Linearly dependent.

In other words even if some v_i's are Linearly independent the WHOLE set is considered LINEARLY DEPENDENT //

ex) Take $\mathbb{R}^2 = V$, consider 2 vectors $u, v \in V$. Each vector has magnitude & direction. Compose $\alpha_1 u + \alpha_2 v = 0$ and think what values of α_1 and α_2 lead to u & v LI

a) Consider u v

b) Consider u v, u & v parallel. //