SINGULAR VALUE DECOMPOSITION (SVD)

Notes taken from Gil Strang's Linear Algebra book.

SVD generalizes the idea of diagonalization to $A \in \mathbb{F}^{m \times n}$:

Consider a symmetric matrix. Then, we can find

$$A = QDQ^T \quad \text{with} \quad Q^TQ = I, \text{ i.e. } Q^T = Q^{-1}!$$

For A symmetric, Q on the left and Q^T on the right are orthogonal.

Is there a decomposition that comes close, but works on $A \in \mathbb{F}^{m \times n}$ generally?

Yes, but we relax the requirement that in $A = BDC$, B & C are orthogonal to each other. We keep insisting that D is diagonal, however:

We show now how to construct a decomposition

$$\text{(SVD)} \qquad A = U_1 \Sigma U_2^*$$

Σ is a matrix with diagonal entries, U_1 & U_2 are unitary, but $U_1^* \neq U_2$, $U_2^* \neq U_1$ in the general case.

The non-zero (diagonal) entries of Σ, matrix are $\sigma_1, \sigma_2, \ldots, \sigma_r$ are all positive. The rank$(A) = r$.

The key to constructing SVD factorization is to work with AA^T and A^TA. Note that AA^T and A^TA are both square matrices.

If $A \in \mathbb{R}^{m \times n}$ then SVD reduces to

$$A = Q_1 \Sigma Q_2^T.$$

Q_1 & Q_2 are not (generally) orthogonal to each other, but $Q_i = Q_i^{-1}$, $i = 1, 2$.

The structure of $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 \ddots & \\ 0 & & \sigma_r \\ \hline & 0 & \end{pmatrix}$,

* $Q_1 \in \mathbb{R}^{m \times m}$ has columns with e'vectors of $AA^T \in \mathbb{R}^{m \times m}$.

* $Q_2 \in \mathbb{R}^{n \times n}$ has columns with e'vectors of $A^TA \in \mathbb{R}^{n \times n}$.

* The singular values on the diagonal of $\Sigma \in \mathbb{R}^{m \times n}$ are the square roots of the nonzero eigenvalues of AA^T (which are the same as e'values of A^TA).