UNITARY OPERATORS, POSITIVE OPERATORS
THE QR DECOMPOSITION:

Ch 7C, LADR

An operator $T : \mathcal{L}(V)$ is **isometric**

iff it preserves the norm.

$$\|Tx\| = \|x\| \quad x \in V$$

Thm: $T : X \to Y$ is isometric iff it preserves the inner product

$$\langle x, y \rangle = \langle Tx, Ty \rangle \quad \forall x, y \in V$$

Lemma: An operator $U : X \to Y$ is an isometry iff $U^* U = I$.

Pf: $\langle x, x \rangle = \langle U^* U x, x \rangle = \langle Ux, U^{**} x \rangle$

$\|x\|$ $\qquad = \|Ux\|$

A unitary matrix with real entries is an orthogonal matrix.

Properties of Unitary Operators

1. $U^* = U^{-1}$
2. if U is unitary, so is U^{-1}
3. A product of unitary operators is unitary.
4. $\det U = \pm 1$ i.e. e'values of U have $|\lambda| = 1$,
5. if U is isometric & $\{v_i\}_{i=1}^{n}$ are orthonormal vectors $\Rightarrow$ $Uv_1, Uv_2, Uv_3 \dots, Uv_n$ is an orthonormal set of vectors.

   If U is unitary then Uv_i $i=1,\dots,n$ form an orthonormal basis.

The QR Factorization $A = QR$

$A \in \mathbb{F}^{m \times n}$

Return to G.S. Recall that

$$Qx = b = x_1 q_1 + x_2 q_2 \cdots x_n q_n$$

here $\langle q_i, q_j \rangle = \delta_{ij}$ and $Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{m \times n}$

so $x_i = (q_i^T b) q_i$ are the coordinates //

Let's find that P in LSQ is $P = QQ^T$: to show this, let

$Qx = b$ (this may be true, but not generally true).

Project b onto colums of Q:

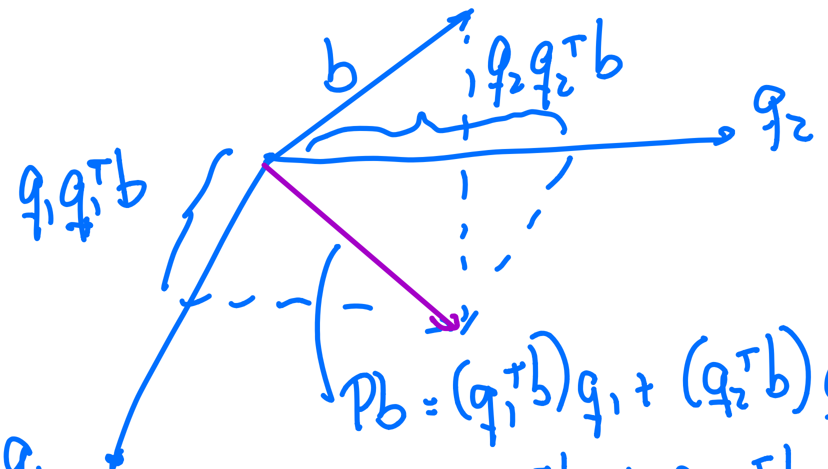
$Pb = Q\hat{x}$ (projection of b onto col(Q))

$Q^T \underbrace{Q\hat{x}}_{Pb} = Q^T b \quad \therefore$

$Pb = QQ^T b \quad \Rightarrow \quad \boxed{P = QQ^T}$

So, geometrically, in $\mathbb{R}^3$:

let $b \in \mathbb{R}^3$ and q_1 & q_2 standard basis of x-y plane:



$$Pb = (q_1^T b)\, q_1 + (q_2^T b)\, q_2$$
$$= q_1 q_1^T b + q_2 q_2^T b$$
$$= QQ^T b$$

. Explicitly,

ex) $q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$Pb = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad q_1^T b\, q_1 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \quad q_2^T b\, q_2 = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$

used G.S. to find q_1, q_2, q_3:

$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. By inspection, $q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$P = q_1 q_1^T + q_2 q_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow Pb = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad //$$

i.e. $P = QQ^T$ projects b onto xy plane.

QR Factorization:

S'pose $A = \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} \in \mathbb{R}^{3\times 3}$, so it is easy to follow:

Here a, b, c are LI

Use G.S to find $q_1\ q_2\ q_3$:

$$a = (q_1^T a) q_1$$

$$b = (q_1^T b) q_1 + (q_2^T b) q_2$$

$$c = (q_1^T c) q_1 + (q_2^T c) q_2 + (q_3^T c) q_3$$

$$A = \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$A \quad = \quad Q \qquad\qquad R$$

(Right of Diagonal)

Rmk: diag(R) are lengths a', b', c'

offdiag(R) lengths of $q_1\ q_2\ q_3$

ex) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

Confirm this yourself numerically

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} R = QR$$

Every matrix $A \in \mathbb{F}^{m \times n}$ with LI columns can be factored $A = QR$. Q is orthonormal & R is upper triangular and invertible.

When $m = n$ Q becomes an orthogonal matrix.

Application of QR to LSQ:

$Ax = b$ (may not have a solution)

settle for LSQ solution $\hat{x}$. Given by

$\hat{x} = (A^TA)^{-1}A^Tb$, assume $A = QR$

Note: $A^TA = (QR)^T(QR) = R^TQ^TQR = R^TR$

if $A^TA\hat{x} = A^Tb$

$R^TR\hat{x} = R^TQ^Tb \Rightarrow \boxed{R\hat{x} = Q^Tb}$

look at $Pb = QR\left[(QR)^T QR\right]^{-1}(QR)^T b$

$$= QR\,(R^TR)^{-1}(QR)^T b$$

$$= Q\underbrace{RR^{-1}}_{I}\underbrace{(R^T)^{-1}R^T}_{I}Q^T b = QQ^T b$$

$\therefore$ $\boxed{Pb = QQ^T b}$

$\Rightarrow$ if $A = QR$ $\Rightarrow$ LSQ can be computed very efficiently, i.e. $O(N)$ where N is the "size" of matrix //

Positive Operators

An operator $T \in \mathcal{L}(V)$ is called positive if T is S.A. and

$$\langle Tv, v\rangle \geq 0 \quad \forall v \in V \quad //$$

Positive definite if $\langle Tv, v\rangle > 0$

Positive indefinite if $\langle Tv, v\rangle \geq 0$

The following are equivalent:

(a) T is positive

(b) T is SA & all e'values of T are non-negative.

(c) T has a positive "square root" i.e. $R^2 = T$

(d) T has a S.A. square root

(e) $\exists R \in \mathcal{L}(V)$ s.t. $T = R^*R$ //

Singular Values of T: S'pse $T \in \mathcal{L}(V)$. The singular values of T are eigenvalues of $\sqrt{T^*T}$. i.e. the singular values are the non-negative square roots of the e'values of T^*T.