Application: Finding approximations to differential equations via Finite-Differences

We want to approximate the linear differential equation

$$\mathcal{L} y = f(x)$$

plus boundary conditions on $y(x)$.

let's take $\mathcal{L} = \frac{d^2}{dx^2}$ and $f(x) = 0$

D.E. $\frac{d^2 y}{dx^2} = 0 \qquad 0 < x < 1$

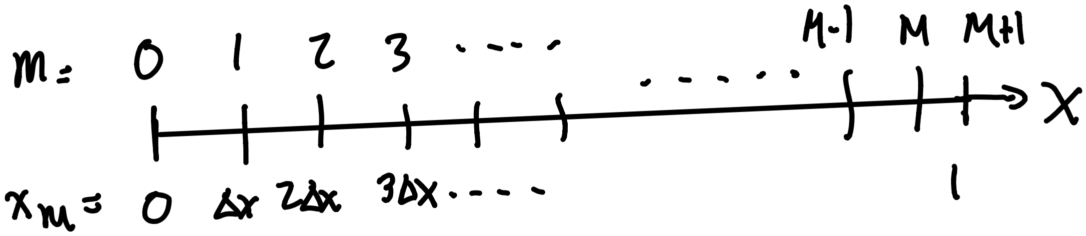
B.C. $y(0) = 1, \; y(1) = 1/2$

(clearly, we can solve this problem exactly by analytical means: integrate twice $y(x) = ax + b$, then use B.C.

$y(0) = a\,0 + b = 1 \Rightarrow b = 1$

$y(1) = a\,1 + b = \frac{1}{2} \Rightarrow a = -\frac{1}{2}$ $\qquad \therefore y = -\frac{1}{2}x + 1$

Using finite differences: first, let's create a lattice $\{x_m\}_{m=0}^{M+1}$ over $0 \le x \le 1$:



$$\Delta x = \frac{1}{M+1} \quad \text{so} \quad x_m = m\Delta x \quad , m = 0, 1 \ldots, M+1$$

We'll seek $u(x_m) \approx y(x_m)$, u is an approximation to y at $x = x_m$ //

$$\text{So} \begin{cases} y(x_m + \Delta x) = y(x_{m+1}) \equiv y_{m+1} \\ y(x_m) = y(x_m) \equiv y_m \\ y(x_m - \Delta x) = y(x_{m-1}) \equiv y_{m-1} \end{cases}$$

Assume $y(x)$ is continuous, and derivatives continuous, and h is small

$$y(x \pm h) = y(x) \pm h y'(x) + \frac{1}{2} h^2 y''(x) \pm \frac{1}{6} h^3 y''' + \frac{1}{24} h^4 y^{IV} + \ldots$$

(via Taylor series)

So

$$y_{m\pm 1} = y_m \pm \Delta x (y')_m + \frac{1}{2}\Delta x^2 (y'')_m \pm \frac{\Delta x^3}{6}(y''')_m + \frac{1}{24}\Delta x^4 (y^{iv})_m + \cdots$$

So

$$y_{m+1} - y_{m-1} = 2y_m + \Delta x^2 (y'')_m + \frac{1}{12}\Delta x^4 (y^{iv}) + \cdots$$

if $\Delta x \ll 1 \Rightarrow$ omit Δx^4 terms and above

$$y_{m+1} - y_{m-1} \approx 2y_m + \Delta x^2 (y'')_m$$

$$\therefore \quad y''(x_m) \approx \frac{y_{m+1} - y_{m-1} - 2y_m}{\Delta x^2}$$

This is a divided-difference approximation to the second derivative. The error is bounded by $\frac{1}{12}\Delta x^2 \max\limits_{0 \le x \le 1} |y^{iv}(x)|$.

The idea is to approximate the solution of

D.E. $y'' = 0$

B.C. $y(0) = 1, \; y(1) = \frac{1}{2}$

on the lattice by

F.D.E. $\frac{1}{\Delta x^2}\left[u_{m+1} + u_{m-1} - 2u_m\right] = 0$

F.B.C. with $u_0 = 1$ and $u_M = \frac{1}{2}$

let $\tilde{U} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M+1} \end{bmatrix} = \begin{bmatrix} u_0 \\ U \\ u_{M+1} \end{bmatrix}$ where $U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}$

we know $u_0 = 1$ and $u_{M+1} = \frac{1}{2}$

So we write F.D.E. & F.B.C. as

$(\star)\quad AU - F = 0$

where

$$A = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & & 1 \\ & & & 1 & -2 \end{bmatrix}$$

What is F in $(\star)$? Look at $m=1$, $m=M$ rows of F.D.E. and F.B.C.

$$\frac{1}{\Delta x^2}\left[-2u_1 + u_2 + u_0\right] = 0 \quad \text{for } m=1.$$

B.C. at $y(0)=1$

$$\text{so } \frac{1}{\Delta x^2}\left[-2u_1 + u_2\right] + \frac{1}{\Delta x^2} = 0$$

$$\frac{1}{\Delta x^2}\left[-2u_M + u_{M+1} + u_{M-1}\right] = 0 \quad \text{for } m=M$$

B.C. $y(1)=\frac{1}{2}$

$$\text{so } \frac{1}{\Delta x^2}\left[-2u_M + u_{M-1}\right] - \frac{1}{2}\frac{1}{\Delta x^2} = 0$$

$\therefore$ Let $F = \begin{bmatrix} \frac{1}{\Delta x^2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\frac{1}{\Delta x^2} \end{bmatrix}$

So an approximation of D.E. + B.C.

is found by solving

$$(\#) \qquad AU = F$$

The solution $U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix} \approx \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}$

and should get better as $M \to \infty$ and $\Delta x \to 0$

(#) is easy solve by an LU factorization.

By induction, you can show that

$$u_m = 1 - \frac{1}{2}\Delta x m \quad , \; m = 0, 1 \ldots, M+1$$