ex) Finding T^* for function spaces.

Let $T: L_2[0,1] \to L_2[0,1]$

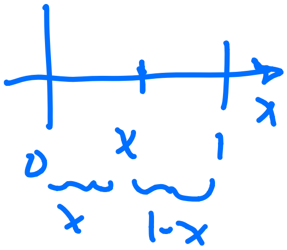
Defined as $Tf(x) \equiv \int_0^x f(s)\,ds, f \in \mathbb{R}$.

$\langle u,v\rangle \equiv L_2[0,1] = \int_0^1 u(x)v(x)\,dx$.

Find T^*: form $\langle Tf,g\rangle = \langle f, T^*g\rangle$

explicitly:

$$\langle Tf,g\rangle = \int_0^1 (Tf)(x)\, g(x)\,dx$$



$$= \int_0^1 \int_0^x f(s)\,ds\, g(x)\,dx$$

$$= \int_0^1 f(s) \int_s^1 g(x)\,dx\,ds = \langle f, T^*g\rangle$$

$\therefore T^* f = \int_x^1 f(s)\,ds$. To see this

intuitively, let $F(x) = \int_0^x f(s)\,ds$ and $G(x) = \int_0^x g(s)\,ds$

i.e. $\frac{dF}{dx} = f, \quad \frac{dG}{dx} = g$

Summarizing Ch7:

For $T: V \to W$, with inner product $\langle \cdot , \cdot \rangle$

the "Adjoint" is T^*, obeying

$$\boxed{\langle Tv, w \rangle = \langle v, T^* w \rangle}$$

$T^*: W \to V$ is a linear map

For a matrix $T \in \mathbb{F}^{m \times n} \Rightarrow T^* = \mathbb{F}^{n \times m}$

if $T \in \mathbb{R}^{m \times n} \Rightarrow T^* = T^T$

SELF ADJOINT : $T = T^*$

$$\boxed{\langle Tv, w \rangle = \langle v, Tw \rangle}$$

e'values of T are the same as T^*
and are real

NORMAL OPERATORS: $[T, T^*] = 0$

i.e. $TT^* - T^*T = 0$

every SA operator is normal.
(since $T = T^*$)

If T is normal $\Rightarrow$ T & T^* have same e'vectors & e'values.

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E'VECTORS OF NORMAL ARE $\perp$

Thm: S'pse $T \in \mathcal{L}(V)$ & is normal. The e'vectors of T corresponding to its distinct e'values are $\perp$.

Pf: S'pse α, β are distinct e'vlues of T associated with e'vectors u, v.

Thus $Tu = \alpha u$, $Tv = \beta v$, $T^*u = \alpha u$, $T^*v = \beta v$

$$\langle Tu, v\rangle - \langle u, T^*v\rangle = (\alpha - \beta)\langle u, v\rangle = 0$$

$$\langle \alpha u, v\rangle - \langle u, \beta v\rangle = 0$$

Since $\alpha \neq \beta$, by assumption,

$$(\alpha - \beta)\langle u, v\rangle = 0$$

$$\Rightarrow \langle u, v\rangle = 0$$

$$\therefore \quad u \perp v$$

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Thm Let T be a linear operator on finite dimension real inner product space V.

T is SA iff ∃ orthonormal basis v for V consisting of the eigenvectors of T.

REAL SYMMETRIC matrices are SA
(SA are normal)

e.g. covariance matrix (in data analysis) //

A Hermitian (or symmetric) matrix is positive definite iff all its eigenvalues are positive.

∴ A complex (real) matrix if positive definite iff its Hermitian (symmetric) part has all positive eigenvalues //

Symmetric Matrices need not be normal. Example, $A = \begin{pmatrix} i & i \\ i & 1 \end{pmatrix}$

$$AA^* = 1+i \neq A^*A = 1-i$$ //

ex) from LADR, an example of T symmetric, not SA:

Let $\mathcal{P}_2(\mathbb{R})$ with inner product

$$\langle p, q \rangle \equiv \int_0^1 p(x)\, q(x)\, dx \qquad (¥)$$

where $p, q \in \mathcal{P}_2(\mathbb{R})$.

Let $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$

defined as $T(a_0 + a_1 x + a_2 x^2) = a_1 x$.

(a) Show that T is not SA.

By inspection

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ so } T^* = T$$ (so Hermitian and symmetric).

However, not SA:

To show this, use $\langle Tu, v \rangle = \langle u, T^* v \rangle$:

$$\left.\begin{aligned} \langle T1, x \rangle &= \langle 0, x \rangle = 0 \\ \langle 1, Tx \rangle &= \langle 1, x \rangle = \tfrac{1}{2} \end{aligned}\right\} \langle T1, x \rangle \neq \langle 1, Tx \rangle$$

Note: $\langle 1, x \rangle = \frac{1}{2}$ $\quad = \frac{1}{2}$ by using (¥)

$\therefore \langle T1, x\rangle \neq \langle 1, Tx\rangle$

$\therefore$ T is not SA //

So do we have a contradiction? NO.

$\{1, x, x^2\}$ are not orthonormal.

We cannot compute the matrix T^* wrt this basis by taking the conjugate transpose of the matrix T. //

THE SPECTRAL THEOREM

Spse $\mathbb{F} \in \mathbb{C}$ and $T \in \mathcal{L}(V)$.

The following are equivalent:

(a) T is normal

(b) V has an ON basis consisting of e'vectors of T

(c) T has a diagonal matrix wrt some ON of V. //

Since $\mathbb{F} = \mathbb{R}$ & $T \in \mathcal{L}(V)$ the following are equivalent:

(a) T is S.A.

(b) V has ON basis consisting of e'vectors of T

(c) T is diagonalizable. //

ex) $A = \begin{pmatrix} 14 & -13 & 8 \\ -3 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix}$ compute ON basis

$$\frac{1}{\sqrt{2}}(1,-1,0), \ \frac{1}{\sqrt{3}}(1,1,1), \ \frac{1}{\sqrt{6}}(1,1,-2)$$

$$D = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -15 \end{pmatrix}$$

$A = SDS^{-1} = A^*$ since D real and $S^{-1} = S^T$

ON Matrix //

ex) Use inner products and adjointness to solve a differential equation.

Solve $\begin{cases} -\mathcal{L}y = f(x) \,, & \text{for } 0 \leq x \leq \ell \,, \\ y(0) = y(\ell) = 0 . & \text{Here } f(x) \text{ is known} \end{cases}$

$$\mathcal{L} \equiv \frac{d^2}{dx^2} + k^2 \qquad k \text{ is a constant}$$

$$L_2[u,v] \equiv \langle u, v \rangle \equiv \int_0^{\ell} u(x)\, v(x)\, dx$$

u, v are continuous real functions, defined over $0 \leq x \leq \ell$, with $u, v = 0$ at $x = 0$ and $x = \ell$.

The general solution is

$$y = y_h + y_p \,, \text{ where}$$

$$-\mathcal{L} y_h = 0 \,,$$

$$-\mathcal{L} y_p = f(x) .$$

The "boundary conditions" are $y(0) = y(\ell) = 0$, mean that the solution sought is zero at $x = 0$, and $x = \ell$. We are requiring this for $f(x)$ as well.

Work with $\mathcal{L}y_h = 0$:

$$\langle y_1, \mathcal{L}y_2 \rangle = \int_0^\ell y_1 y_2'' \, dx + \int_0^\ell y_1 k^2 y_2 \, dx$$

$$\langle \mathcal{L}y_1, y_2 \rangle = \int_0^\ell y_1'' y_2 \, dx + \int k^2 y_1 y_2 \, dx$$

$$\langle y_1, \mathcal{L}y_2 \rangle - \langle \mathcal{L}y_1, y_2 \rangle = \int_0^\ell y_1 y_2'' \, dx - \int_0^\ell y_1'' y_2 \, dx$$

[integrate by parts:

let $u = y_1$ $v = y_2'$

$du = y_1' dx$ $dv = y_2'' dx$]

$$= -\int_0^\ell y_1' y_2' \, dx + \int_0^\ell y_1' y_2' \, dx + y_1 y_2' \Big|_0^\ell - y_1' y_2 \Big|_0^\ell$$

$$\langle y_1, \mathcal{L}y_2 \rangle - \langle \mathcal{L}y_1, y_2 \rangle = y_1 y_2' \Big|_0^\ell - y_1' y_2 \Big|_0^\ell$$

$$= y_1(\ell) y_2'(\ell) - y_1(0) y_2'(0) - y_1' y_2 \Big|_0^\ell$$

$= 0$ by boundary conditions!

This calculation proves that $\mathcal{L}$ is SA ∴ has ON basis. In fact, we'll see ON and distinct e'values!

Compute space for $\mathcal{L}$:

$-\mathcal{L}y = \lambda y$ is the e'value problem.

e'values $\lambda_n = \left(\frac{n\pi}{\ell}\right)$ $n = 1, 2, \ldots$

e'vectors $\varphi_n = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right)$ $n=1,2\ldots$

Note that: $\int_0^\ell \varphi_n^2 \, dx = 1$ (show by direct calculation)

Also Note: $\int_0^\ell \varphi_n \varphi_m \, dx = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$

.c.

which confirms that $\{\varphi_n\}_{n=1}^\infty$ are ON basis.

$y_h = \sum_{n=1}^\infty \alpha_n \varphi_n$ but $\mathcal{L} y_h = 0$ and $y(0) = y(\ell) = 0 \Rightarrow \alpha_n = 0$ $n=1,2\ldots$

let $y_p = \sum_{n=1}^\infty \beta_n \varphi_n$ (*) β_n constants, to be determined.

$-\mathcal{L} y_p = f(x)$. Using (*),

$$(\dagger) \quad -\mathcal{L} \sum_{n=1}^\infty \beta_n \varphi_n = f(x).$$

let $f(x) = \sum_{n=1}^\infty \gamma_n \varphi_n$. (∃)

to find γ_n, use orthogonality of $\{\varphi_n\}_{n=1}^\infty$:

$$\int_0^\ell \varphi_m(x) f(x) \, dx = \int_0^\ell \varphi_m(x) \sum_{n=1}^\infty \gamma_n \varphi_n \, dx$$

$$= \sum_{n=1}^\infty \gamma_n \overbrace{\int_0^\ell \varphi_n(x) \varphi_m(x) \, dx}^{= \delta_{nm}}$$

(switched Σ and $\int$).

$$\therefore \int_0^\ell \varphi_n(x) f(x)\,dx = \gamma_n , \quad n=1,2,\dots .$$

So we can find all of the γ_n, since we know $f(x)$.

Now Go back to (7) and use (☆) and (∃):

$$-\mathcal{L}\sum_{n=1}^{\infty} \beta_n \varphi_n = \sum_{n=1}^{\infty} \gamma_n \varphi_n . \quad (\$)$$

if we find β_n we know y_p. Write (\$) as

$$-\sum_{n=1}^{\infty} \beta_n \mathcal{L} \varphi_n = \sum_{n=1}^{\infty} \gamma_n \varphi_n$$

or $$\sum_{n=1}^{\infty} \left(-\beta_n \mathcal{L} - \gamma_n\right) \varphi_n = 0 . \quad (\#)$$

Multiply both sides of (#) by φ_m and integrate from 0 to ℓ:

Exchanging Σ and $\int$:

$$\sum_{n=1}^{\infty} \int_0^\ell \varphi_m \left[-\beta_n \mathcal{L} \varphi_n - \gamma_n \varphi_n\right] dx = 0 , \text{ or}$$

$$\sum_{n=1}^{\infty} \int_0^\ell \varphi_m \left[-\beta_n \mathcal{L} \varphi_n\right] dx = \gamma_n \quad (₡)$$

since $-\mathcal{L}\varphi_n = \lambda_n \varphi_n$

$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n=1,2\dots$

Replace $-\mathcal{L}\varphi_n = \lambda_n \varphi_n$ in (#) to get

$$\sum_{n=1}^{\infty} \int_0^l \varphi_m \left[\beta_n \lambda_n\right] \varphi_n \, dx = \gamma_n \qquad n = 1, 2 \ldots n$$

Exchange Σ and $\int$:

$$\sum_{n=1}^{\infty} \beta_n \lambda_n \int_0^l \overset{\delta_{nm}}{\overset{\shortparallel}{\varphi_m \varphi_n}} \, dx = \gamma_n \, , \; n = 1, 2 \ldots$$

or $\beta_n \lambda_n = \gamma_n \Rightarrow \beta_n = \dfrac{\gamma_n}{\lambda_n}$

$n = 1, 2 \ldots$

$$\therefore \; y_p = \sum_{n=1}^{\infty} \frac{\gamma_n}{\lambda_n} \varphi_n(x)$$

$$\gamma_n = \int_0^l f(x) \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right) dx \qquad n = 1, 2 \ldots$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad n = 1, 2 \ldots$$

$$\varphi_n = \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right)$$

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