ADJOINT OPERATORS, SELF ADJOINT NORMAL OPERATORS

LADR Ch 7

The ADJOINT of an operator T is denoted by

T^*.

Spse $T \in \mathcal{L}(V, W)$. The adjoint of T is the operator $T^*: W \to V$ st for $v \in V$, $w \in W$ (V and W could be one & the same) will have the property

(‡) $\langle Tv, w \rangle = \langle v, T^* w \rangle$

(this is a way to find T from T^* or T^* from T)

for every v & every w

(not all linear operators have an adjoint, or be adjoint on all v, w)

ex) $T: \mathbb{R}^3 \to \mathbb{R}^2$

Given as $T(x_1, x_2, x_2) = (x_2 + 3x_3, 2x_1)$.

Find T^*. It must be that $T^*: \mathbb{R}^2 \to \mathbb{R}^3$, if it exists.

USE (‡) to find T^*:

Pick $(y_1, y_2) \in \mathbb{R}^2$. Then

$$\langle (x_1, x_2, x_3), T^*(y_1, y_2) \rangle = \langle T(x_1, x_2, x_3), (y_1, y_2) \rangle$$

this is

$$\langle v, T^* w \rangle = \langle Tv, w \rangle$$

$$= \langle (x_2 + 3x_3, 2x_1), (y_1, y_2) \rangle$$

$$= x_2 y_1 + 3x_3 y_1 + 2x_1 y_2$$

$$= \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1) \rangle$$

$\therefore$ $T^*(y_1, y_2) = (2y_2, y_1, 3y_1)$. Explicitly,

$$T^* \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_2 \\ y_1 \\ 3y_1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_2 \\ y_1 \\ 3y_1 \end{pmatrix} \Longrightarrow \begin{cases} ay_1 + by_2 = 2y_2 \\ cy_1 + dy_2 = y_1 \\ ey_1 + fy_2 = 3y_1 \end{cases}$$

$a=0 \quad b=2$

$c=1 \quad d=0$

$e=3 \quad f=0$

$$T^* = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 3 & 0 \end{pmatrix}.$$ By inspection.

To find T:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + 3x_3 \\ 2x_1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2+3x_3 \\ 2x_1 \end{pmatrix} \Rightarrow \begin{matrix} ax_1 + bx_2 + cx_3 = x_2 + 3x_3 \\ dx_1 + ex_2 + fx_3 = 2x_1 \end{matrix}$$

let $e = f = 0$

$d = 2$

$a = 0$

$b = 1$

$c = 3$

$$T = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

We note here that $T^* = T^T$

ex) Let $u \in V$, $x, w \in W$

Let $T \in \mathcal{L}(V, W)$ defined as

$$Tv = \langle v, u \rangle x \quad \forall v \in V.$$

Find T^*:

Fix $w \in W$, for $v \in V$

$\langle v, T^* w\rangle = \langle Tv, w\rangle$

$= \langle \langle v, u\rangle x, w\rangle$

$=$

$= \langle v, u\rangle \langle x, w\rangle$

$= \langle v, \langle w, x\rangle u\rangle$

$\therefore T^* w = \langle w, x\rangle u$ //

Rmk: The adjoint is a linear map

If $T \in \mathcal{L}(V, W)$ then $T^* \in \mathcal{L}(W, V)$

PROPERTIES OF ADJOINT OPERATORS

$(S+T)^* = S^* + T^*$

$(\lambda T)^* = \bar{\lambda} T^*$, $\lambda \in \mathbb{F}$

$(T^*)^* = T$

$I^* = I$

$(ST)^* = T^* S^*$

where $T: \mathcal{L}(U, V)$ $T^*: \mathcal{L}(V, U)$

GEOMETRY OF SUBSPACES

$T \in \mathcal{L}(V,W)$	$A \in \mathbb{F}^{m\times n}$
$\text{Null}(T^*) = (\text{range } T)^\perp$	$\text{Null}(A^*) = (\text{Col}(A))^\perp$
$\text{range}(T^*) = (\text{null } T)^\perp$	$\text{Col}(A^*) = (\text{null}(A))^\perp$
$\text{null}(T) = \text{range}[(T^*)^\perp]$	$\text{null}(A) = (\text{Col}(A^*))^\perp$
$\text{range}(T) = \text{null}[(T^*)^\perp]$	$\text{Col}(A) = \text{null}[(A^*)^\perp]$

SELF ADJOINT (HERMITIAN)

if $T: \mathcal{L}(V)$ and $T = T^*$ then

T is called self adjoint

Hence,

$$(\maltese) \quad \begin{bmatrix} \langle Tv, w\rangle = \langle v, Tw\rangle \\ v, w \in V \end{bmatrix}$$

ex) $T = \begin{pmatrix} 2 & b \\ 3 & 7 \end{pmatrix}$ on $\mathbb{R}^2$, $\langle u, v\rangle = u^T v$, $u, v \in \mathbb{R}^2$.

Find b s.t. $T = T^*$

Using $(\maltese)$ $\left\langle T\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, T\begin{pmatrix} w \\ z \end{pmatrix}\right\rangle$

or $(2x+by)w + (3x+7y)z$

$$= x(2w+bz) + 3yw + 7yz$$

$$\Rightarrow b = 3$$

$$\therefore \quad T = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}; \; T^* = T^T = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$$

If T is a matrix and $T^* = T$ we say that T is Hermitian. Here, $T \in \mathbb{F}^{n \times n}$.

Thm: All eigenvalues of a Hermitian matrix must be real

Pf: Suppose $T = T^*$ on $v \in V$. Let λ be an eigenvalue of T, $\lambda \in \mathbb{F}$.

$Tv = \lambda v$, v the eigenvector associated with λ.

then $\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda \|v\|^2$.

$$\Rightarrow \lambda \|v\|^2 = \langle \lambda v, v\rangle = \langle Tv, v\rangle$$

$$\langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\|v\|^2$$

$$\Rightarrow \lambda = \bar{\lambda} \text{ or } \lambda \text{ is real.}$$

Normal Operators

S'pse $T \in \mathcal{L}(V)$. If $TT^* = T^*T$

i.e. $[T, T^*] = 0$, the commutator is 0, $\Rightarrow$ T is normal

Every self-adjoint operator is 'normal,'
since $T = T^*$

Th'm: T is normal iff $\|Tv\| = \|T^*v\|$
$\forall v \in V$.

Pf: if T normal $\Rightarrow TT^* = T^*T$, by definition.

$\Leftrightarrow \langle (TT^* - T^*T)v, v \rangle = 0 \quad \forall v \in V$

$\Leftrightarrow \langle T^*Tv, v \rangle = \langle TT^*v, v \rangle \quad \forall v \in V$

$\Leftrightarrow \|Tv\|^2 = \|T^*v\|^2$ //

If T is normal $\Rightarrow$ T & T^* have same eigenvectors

$0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^* v\|$

$= \|(T^* - \bar{\lambda} I)v\|$ $\therefore$ v is the e'vector of T^* and