Orthonormal Matrices in $\mathbb{F}^n$

It's a matrix (square or rectangular) composed of n LI column vectors $\underset{\sim}{q}_i$, $i=1,2,..,n$, all $\perp$, all $\|q_i\|=1$:

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$ the orthonormal matrix

$$\|q_i\| = 1 \qquad \langle q_i, q_j \rangle = \delta_{ij} \begin{cases} =1 & \text{if } i=j \\ =0 & \text{otherwise} \end{cases}$$

$$\boxed{Q^T Q = I}$$ true for $n\times n$ & $m\times n$

$$\boxed{Q Q^{-1} = I}$$ true for $n\times n$

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Another important property of Orthonormal matrices is that they "preserve length":

To see this:

let $x, y \in V$ and Q be an orthonormal matrix. Then

$$\langle Qx, Qy \rangle = x^T Q^T Q y = x^T I y = \langle x, y \rangle$$

in particular, if $x = y$:

$$\|Qx\| = \|x\|$$

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Since Q is made up of basis of V, and q_i are orthonormal (LI & complete) in V, also suppose $b \in V$, then

$$b = x_1 q_1 + x_2 q_2 \cdots x_n q_n .$$

Since $q_i^T q_i = 1$, we can find x_i easily:

$$q_i^T b = x_i q_i^T q_i = x_i \quad i = 1, 2 \ldots, n.$$

More generally

$$b = (q_1^T b) q_1 + \cdots + (q_n^T b) q_n$$

$$x = Q^T b$$

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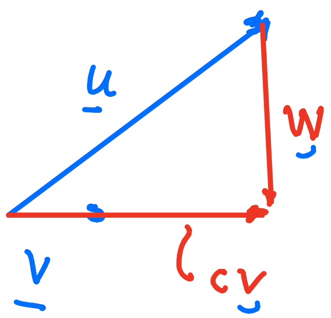
Recall



$$\underset{\sim}{u} = c\underset{\sim}{v} + \underset{\sim}{w}$$

$$c = \frac{\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle}{\|\underset{\sim}{v}\|^2}$$

$$\underset{\sim}{w} = \underset{\sim}{u} - \frac{\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle}{\|\underset{\sim}{v}\|^2} \underset{\sim}{v} .$$

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Now, suppose 3 vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ independent are known.

Want to produce q_1, q_2, q_3 are $\perp$ and normalized.

① pick $\underset{\sim}{a}$: can choose $\underset{\sim}{q_1} \parallel \underset{\sim}{a}$. So,

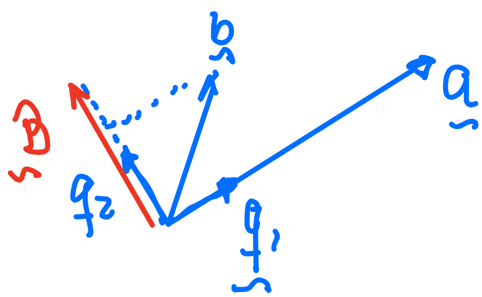
$$\underset{\sim}{q_1} = \frac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|} .$$

② To find $\underset{\sim}{q_2}$: if $\underset{\sim}{b}$ has a component in the $\underset{\sim}{q_1}$ direction, we need to subtract it:

$\underset{\sim}{B} = \underset{\sim}{b} - (\underset{\sim}{q}_1^T \underset{\sim}{b})\, \underset{\sim}{q}_1$, normalizing

$$\underset{\sim}{q}_2 = \frac{\underset{\sim}{B}}{\|\underset{\sim}{B}\|}.$$

so $\underset{\sim}{B} \perp \underset{\sim}{a}$ and $\Rightarrow \underset{\sim}{q}_1 \perp \underset{\sim}{q}_2$, with $\|\underset{\sim}{q}_2\| = 1$.



③ Find $\underset{\sim}{q}_3$: use $\underset{\sim}{c}$, but subtract from it whatever is in the $\underset{\sim}{q}_1$ & $\underset{\sim}{q}_2$ directions:

$\underset{\sim}{C} = \underset{\sim}{c} - (\underset{\sim}{q}_1^T \underset{\sim}{c})\, \underset{\sim}{q}_1 - (\underset{\sim}{q}_2^T \underset{\sim}{c})\, \underset{\sim}{q}_2$. Normalize

$$\underset{\sim}{q}_3 = \frac{\underset{\sim}{C}}{\|\underset{\sim}{C}\|}.$$

(GS) Gram-Schmidt Process generalizes this to handle the $\mathbb{F}^n$ case:

Let's do a concrete case first: Given

ex) $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \underset{\sim}{c} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$,

$$\underset{\sim}{q}_1 = \frac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|} = \frac{\underset{\sim}{a}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\underset{\sim}{B} = \underset{\sim}{b} - (\underset{\sim}{q}_1^T \underset{\sim}{b})\underset{\sim}{q}_1,$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\underset{\sim}{q}_2 = \frac{\underset{\sim}{B}}{\|\underset{\sim}{B}\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}.$$

$$\underset{\sim}{C} = \underset{\sim}{c} - (\underset{\sim}{q}_1^T \underset{\sim}{c})\underset{\sim}{q}_1 - (\underset{\sim}{q}_2^T \underset{\sim}{c})\underset{\sim}{q}_2,$$

$$\underset{\sim}{G} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underset{\sim}{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If asked to construct the orthonormal matrix for these

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}.$$

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G-S: Given $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$, all LI:

for $j = 1:n$

$$\underset{\sim}{A}_j = \underset{\sim}{a}_j - \sum_{i=1}^{j-1} \underset{\sim}{q}_i^T \underset{\sim}{a}_j \underset{\sim}{q}_i$$

$$\underset{\sim}{q}_j = \frac{\underset{\sim}{A}_j}{\|\underset{\sim}{A}_j\|}$$

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Ex) Find an orthonormal basis for $\mathbb{P}_2(\mathbb{R})$, with inner product:

$$\langle p, q \rangle = \int_{-1}^{1} pq \, dx,$$

$\mathcal{P}_2(\mathbb{R})$ is spanned by $\{1, x, x^2\}$. Choose

$$q_1 = \frac{1}{\|1\|}, \quad \|1\|^2 = \int_{-1}^{1} 1^2 dx \quad \therefore \|1\| = \sqrt{2}.$$

$$q_1 = \sqrt{\tfrac{1}{2}}.$$

$$B = x - \langle x, q_1 \rangle q_1 = x - \int_{-1}^{1} x\sqrt{\tfrac{1}{2}}\,dx \sqrt{\tfrac{1}{2}}$$

$$= x$$

$$\|B\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}.$$ Then

$$q_2 = \frac{B}{\|B\|} = \sqrt{\tfrac{3}{2}}\,x.$$

$$C = x^2 - \langle x^2, q_1 \rangle q_1 - \langle x^2, q_2 \rangle q_2$$

$$= x^2 - \int_{-1}^{1} x^2 \sqrt{\tfrac{1}{2}}\,dx \sqrt{\tfrac{1}{2}} - \int_{-1}^{1} x^2 \sqrt{\tfrac{3}{2}}\,x\,dx \sqrt{\tfrac{3}{2}}\,x$$

$$= x^2 - 1/3.$$ Hence

$$q_3 = \frac{c}{\|c\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right), \text{ since}$$

$$\|c\|^2 = \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx = \frac{8}{45}. \quad //$$

The ORTHOGONAL PROJECTION:

Let $\{v_i\}_{i=1}^{n}$ be orthonormal basis for V, a vector space.

$\Rightarrow$ the orthogonal projection of $\underset{\sim}{w}$ is

$$P_V \underset{\sim}{w} = \sum_{k=1}^{n} \alpha_k v_k, \text{ where}$$

$$\alpha_k = \frac{\langle w, v_k \rangle}{\|v_k\|^2}. \quad //$$

Orthogonal Complement:

$$V = E \oplus E^{\perp}$$

$E^{\perp}$ is the orthogonal complement of E

For a subspace E, its orthogonal complement $E^\perp$ is the set of vectors $\perp$ to E:

$$E^\perp = \{x : x \perp E\}.$$

Note: if $x, y \perp E \Rightarrow \alpha x + \beta y \perp E$

By definition of the orthogonal projection, any vector in an inner product space V admits a unique representation:

$\underset{\sim}{v} = \underset{\sim}{v_1} + \underset{\sim}{v_2}$ where $\underset{\sim}{v_1} \in E$ $\underset{\sim}{v_2} \perp E$

i.e. $\underset{\sim}{v_2} \in E^\perp$.

$\underset{\sim}{v_1} = P_E \underset{\sim}{v}$

Note: $(E^\perp)^\perp = E$

Note: $\{0\} = V^\perp$ and $\{0\}^\perp = V$

$\therefore E \cap E^{\perp} = \{0\}$.

Note: if U & W are subsets of V and $U \subset W$

$\Rightarrow W^{\perp} \subset U^{\perp}$

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