The Norm: Let $v \in V$, a vector space.

For a vector v the **norm** is indicated

$$\|v\|$$

Suppose $v \in \mathbb{F}^n$, i.e.

$$v = (v_1, v_2, \dots, v_n)^T, \text{ then}$$

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2}$$ "the 2-norm"

$$\|v\|_p = \left[\sum_{i=1}^{n} |v_i|^p\right]^{1/p}$$ "the p-norm"

$p = 1, 2, \dots, \infty$. Note that

$$\|v\|_\infty = \max_{1 \le i \le n} |v_i|$$

Suppose $v \in V$

$$\begin{cases} \|v\| = 0 & \text{iff } v = \emptyset \\ \|\lambda v\| = |\lambda| \, \|v\| & \forall \lambda \in \mathbb{F} \end{cases}$$

ORTHOGONALITY

Two vectors $u, v \in V$ are orthogonal (denoted $u, v \perp$) if $\langle u, v \rangle = 0$. //

ex) Consider the space of continuous functions V consisting of functions, defined for $x \in [-1, 1]$, say.

if $\int_{-1}^{1} f \bar{g} \, dx = 0$ then f and g are $\perp$

ex) A more familiar example: $u, v \in \mathbb{R}^n$

$$u^T v = 0 \quad \text{or} \quad u \cdot v = 0$$

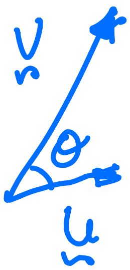
$\Rightarrow u \,\&\, v \perp$. //

Orthogonality & $\emptyset$:

(a) $\emptyset$ is orthogonal to every vector in V

(b) $\emptyset$ is the only vector in V that's $\perp$ to itself.

Triangle Inequality



$\langle u,v\rangle = u^T v = \|u\|\|v\|\cos\theta$

if $\langle u,v\rangle = 0$
then $u,v \perp$ $(u,v \neq 0)$

//

Let $u,v \in \mathbb{F}^n$. Let's compute

$\|u+v\|_2^2$ ← let's omit this subscript in what follows.

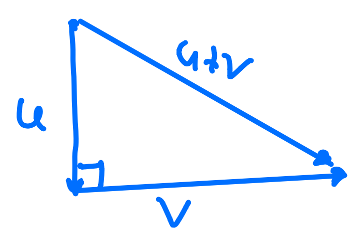
$= \langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle$

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\mathrm{Re}(\langle u,v\rangle).$

Pythagoras is obtained if $\langle u,v\rangle = 0$

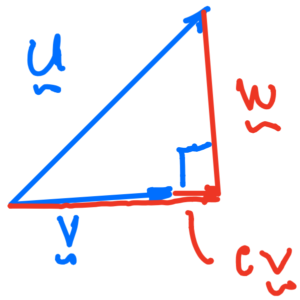
$\Rightarrow \|u+v\|_2^2 = \|u\|_2^2 + \|v\|_2^2$

when $u,v \perp$



//

A useful construct:



Want to write u in terms of a scalar multiple of v and some w s.t. $cv \perp w$:

(this is fundamental to Gram Schmidt Algorithm)

$$u = cv + w = cv + (u - cv)$$

all we need to find is the value c such that v and $(u - cv)$ are $\perp$

$$0 = \langle v, u - cv \rangle = \langle v, u \rangle - c\langle v, v \rangle$$

$$\therefore \quad c = \frac{\langle v, u \rangle}{\|v\|^2}$$

$$\therefore \quad u = v + (u - cv) = v + \overbrace{u - \frac{\langle v, u \rangle}{\|v\|^2} v}^{w}$$

"Orthogonal Decomposition"

Cauchy Schwarz Inequality:

S'pse $\underline{u}, \underline{v} \in V$. Then

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \, \|\underline{v}\|$$

Equal only if $\underline{u}$ (or $\underline{v}$) is a scalar multiple of $\underline{v}$ (or $\underline{u}$)

__Pf__: (a) If $\begin{cases} \underline{v} = 0 \\ \text{or} \\ \underline{u} = 0 \end{cases}$, then $|\langle \underline{u}, \underline{v} \rangle| = 0$.

(b) if $\underline{u} = c\underline{v}$

$$|\langle \underline{u}, \underline{v} \rangle| = c|\langle \underline{v}, \underline{v} \rangle| = c\|\underline{v}\|^2$$

get equality.

(c) Assume neither of the above:

let $\underline{u} = \dfrac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{v}\|^2} \underline{v} + \underline{w}$

where $\underline{w} \perp \underline{v}$

by Pythagoras:

$$\|\underset{\sim}{u}\|^2 = \left\| \frac{\langle u, v\rangle}{\|\underset{\sim}{v}\|^2} v \right\|^2 + \|\underset{\sim}{w}\|^2, \text{ or}$$

$$\|\underset{\sim}{u}\|^2 = \frac{|\langle \underset{\sim}{u}, \underset{\sim}{v}\rangle|}{\|\underset{\sim}{v}\|^2} + \|\underset{\sim}{w}\|^2 \geq \frac{|\langle \underset{\sim}{u}, \underset{\sim}{v}\rangle|^2}{\|\underset{\sim}{v}\|^2} \Rightarrow$$

$\therefore \; \|u\|^2 \|v\|^2 \geq |\langle u, v\rangle|^2$, the C.S. inequality. //

ex) $\underset{\sim}{u} = \sum_{i=1}^{n} \hat{e}_i u_i \qquad \underset{\sim}{v} = \sum_{i=1}^{n} \hat{e}_i v_i$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \sum_{i=1}^{n} u_i v_i \qquad |\underset{\sim}{u} \cdot \underset{\sim}{v}| = \sum_{i=1}^{n} |u_i v_i|$$

$$\|\underset{\sim}{u}\| = \sqrt{\sum_{i=1}^{n} u_i^2} \qquad \|\underset{\sim}{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

$$\|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 = \sum_{i=1}^{n} u_i^2 \sum_{i=1}^{n} v_i^2 \geq \sum_{i=1}^{n} |u_i v_i|^2. \;//$$

ex) If f, g are continuous real functions on $x \in [-1, 1]$:

$$\left| \int_{-1}^{1} f g \, dx \right| \leq \int_{-1}^{1} |f|^2 dx \int_{-1}^{1} |g|^2 dx$$

TRIANGLE INEQUALITY

$$\| \underline{u} + \underline{v} \| \leq \| \underline{u} \| + \| \underline{v} \|$$

equal only when $\underline{u}$ & $\underline{v}$ are non-negative multiples of each other, i.e. $\|\underline{u}\| \|\underline{v}\| = \langle \underline{u} \cdot \underline{v} \rangle$.

<u>Pf:</u> $\| \underline{u} + \underline{v} \|^2 = \langle \underline{u} + \underline{v}, \underline{u} + \underline{v} \rangle^2$

$= \langle \underline{u}, \underline{u} \rangle + \langle \underline{v}, \underline{v} \rangle + \langle \underline{u}, \underline{v} \rangle + \overline{\langle \underline{u}, \underline{v} \rangle}$

$= \| \underline{u} \|^2 + \| \underline{v} \|^2 + 2 \operatorname{Re} \langle \underline{u}, \underline{v} \rangle$

$\leq \| \underline{u} \|^2 + \| \underline{v} \|^2 + 2 \underbrace{\left| \langle \underline{u}, \underline{v} \rangle \right|}_{\text{use CS on this}}$

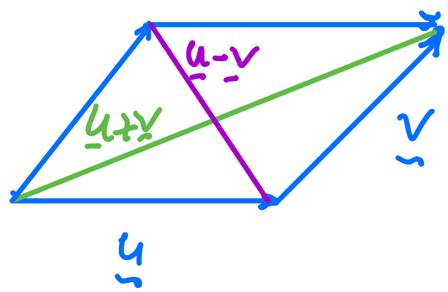
$\leq \| \underline{u} \|^2 + \| \underline{v} \|^2 + 2 \| \underline{u} \| \| \underline{v} \|$

$= \left(\| \underline{u} \| + \| \underline{v} \| \right)^2$

Parallelogram Inequality

S'pse $\underset{\sim}{u}, \underset{\sim}{v} \in V$

$$\|\underset{\sim}{u}+\underset{\sim}{v}\|^2 + \|\underset{\sim}{u}-\underset{\sim}{v}\|^2 = 2\left(\|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2\right)$$



Normed Vector Spaces

A vector space equipped with a norm.

Any inner product space is a normed space. However, there are more normed spaces than inner product spaces if $v \in V$ then $\|v\|$ is a norm if

$\|\alpha v\| = |\alpha| \, \|v\|$ for $\alpha \in \mathbb{F}$

$\|u+v\| \leq \|u\| + \|v\|$

$\|v\| \geq 0$

$\|v\| = 0$ iff $v = \emptyset$.

so, if $\|v\| \equiv \sqrt{v \cdot \bar{v}}$, with $v \in V \Rightarrow V$ is an inner product space & is a norm space.

<u>Thm:</u> A norm in a normed space is obtained from some inner product iff it satisfies the parallelogram equality

Orthogonal Basis (basis that are $\perp$)

Orthonormal Basis: basis $\perp$ and all vectors have norm $= 1$.

e.g. $u = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$u \perp v$ and $\|u\| = 1$, $\|v\| = 1$.

They are orthonormal, but not a basis of $\mathbb{R}^3$

The Orthonormal Basis of V is $\perp$, $\perp$, $\|v\| = 1$

e.g. $V = \mathbb{R}^4$. An orthonormal basis for $\mathbb{R}^4$ is

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}.$$

//

Writing a vector $v \in V$ in terms of the orthonormal basis:

Suppose $\mathcal{B} = \{b_i\}_{i=1}^{n}$, an orthogonal basis for V.

Take $x \in V$ we write

$$x = \sum_{i=1}^{n} \alpha_i b_i, \text{ where } \alpha_i \in \mathbb{F}.$$

To find α_i: project x into subspace spanned by b_i:

That is, take the dot product of x and b_i:

$$b_i \cdot x = \sum_{i=1}^{n} \alpha_i b_i b_i$$

$$\therefore \quad \alpha_i = \frac{b_i \cdot x}{b_i \cdot b_i} \qquad i = 1, 2, \ldots n$$

if $\|b_i\| = 1$, orthonormal,

then $\alpha_i = b_i \cdot x \quad 1 \leq i \leq n.$