Examples (Diagonalization)

ex) The Rotation Matrix in $\mathbb{R}^2$: Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ (1) compute e'values, vectors:

$$\det(A - \lambda I) = 0 = p(\lambda) = \lambda^2 + 1 = 0$$

$\lambda_{1,2} = \pm i$ are the e'values.

$$(A - \lambda_1 I)x_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$(A - \lambda_2 I)x_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ +i \end{bmatrix} = \bar{x}_1.$$

If $z = \alpha + i\beta$
$\bar{z} = \alpha - i\beta$ (complex conjugate)

There are 2 distinct e'vectors, x_1 & x_2.

$A = SDS^{-1}$ For $S = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}$, i.e.

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$ Form $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$

Check on your own: $AS = SD$

ex) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$. Will have 2 distinct e'values, but 3 L.I. eigenvectors $\therefore$ diagonalization is possible

(1) Compute e'values:

$$\det(A - \lambda I) = 0 = (2-\lambda)^2(1-\lambda) = 0$$

$\lambda = 2$, repeated root,

$\lambda = 1$, simple root.

(2) Compute eigenvectors:

For λ_1: $(A - \lambda_1 I)x_1 = 0$ or $x_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

for $\lambda_2 = \lambda_3 = 2$

$$(A - \lambda_2 I)x_j = 0 \quad j = 2,3$$

$$(A - 2I)x_j = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} x_j \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = x_3$ work and are L.I.

so

$$S = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

SOME REMARKS ON $A = SDS^{-1}$

(1) If A has no repeated e'values $\Rightarrow$ it will have the required number of LI e'vectors.

(2) In $A = SDS^{-1}$, the S is <u>not</u> unique, but D is.

(3) $AS = SD$ holds if columns of S are e'vectors of A and not otherwise. Other matrices will not produce the D: this has to do with the rules of matrix multiplication: to see this, suppose first column of S is $v_1 \Rightarrow SD = \lambda_1 v_1$.
If this is to agree with first column of $AS = Av_1$
$\Rightarrow$ v_1 has to be the e'vector associated with λ_1.

(4) Since not all matrices have LI e'vectors not all matrices are diagonalizable. //

An example of a non-diagonalizable matrix:

ex) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda_1 = \lambda_2 = 0$ ($\det(A - \lambda I) = 0$)

so $(A - \lambda I)x = 0 = Ax = 0$, has an eigenvector $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

That's the only e'vector of A.

hence A is not diagonalizable //

Thus far we covered additive and multiplication (algebra).
The next topic brings in the geometric side:

Inner Product Spaces

LADR Ch 7

Vector spaces equipped with an inner product.

Some topics we need to cover first (or along the way):

1. Cauchy-Schwarz Inequality
2. Gram-Schmidt orthogonalization procedure.
3. Linear functionals on inner product (vector) spaces
4. Calculating distances (norm) between subspaces.

What's an inner product?

Inner Product & Norm

S'pse $x \in \mathbb{R}^n$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 \dots x_n^2}$ is the L_2 norm.

S'pse that $x = \begin{pmatrix} x_1 + i y_1 \\ x_2 + i y_2 \\ \vdots \\ x_n + i y_n \end{pmatrix}$ i.e. $x \in \mathbb{C}^n$

How to compute the L_2 norm?

The complex conjugate of the vector x is

$$\bar{x} = \begin{pmatrix} x_1 - iy_1 \\ x_2 - iy_2 \\ \vdots \\ x_n - iy_n \end{pmatrix}$$

so we use an overbar to indicate complex conjugate.

L_2 norm for $x \in \mathbb{C}^n$

$$\|x\|_2 = \sqrt{x^T \bar{x}} = \sqrt{\bar{x}^T x}$$

transpose

We also introduce the Hermitian of a matrix:

Hermitian

For $A \in \mathbb{C}^{n \times m}$, A^* is the Hermitian matrix.

$A^* = \bar{A}^T$. If $A \in \mathbb{R}^{n \times m} \Rightarrow A^* = A^T$

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The dot product for two vector $v, w \in \mathbb{R}^n$

$$v \cdot w \equiv v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

$$v \cdot w = v^T w$$

Properties of the dot product:

$$\begin{cases} x \cdot x \geq 0 \quad \forall x \in \mathbb{R}^n \\ x \cdot x = 0 \quad \text{iff } x = \emptyset \\ x \cdot y = y \cdot x \end{cases}$$

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Inner Product on V is a function that takes an ordered pair (u,v) of elements of V to a number

$\langle u,v\rangle \in \mathbb{F}$. It has

Properties of the inner product:

(a) Positivity: $\langle u,v\rangle \geq 0 \quad \forall\, u,v \in V$

(b) Definiteness: $\langle v,v\rangle = 0$ iff $v = \emptyset$.

(c) Additivity (in the n^{th} slot). For the first slot:

$$\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$$

(d) Homogeneity (in the n^{th} slot)

first slot:

$$\langle \lambda u, v\rangle = \lambda\langle u,v\rangle \quad \lambda \in \mathbb{F}$$

(e) Conjugate Symmetry

$$\langle u,v\rangle = \overline{\langle u,v\rangle} \quad \forall\, u,v \in V$$

ex) Euclidean inner product on $\mathbb{F}^n$:

let $w,v \in \mathbb{F}^n$, the inner product is

$$\langle w,v\rangle = w_1\bar{v}_1 + w_2\bar{v}_2 + \cdots w_n\bar{v}_n$$

ex) Inner product on vector space of (possibly complex) continuous functions on the interval $[-1,1]$:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\bar{g}(x)\,dx$$

if $g \in \mathbb{R}$, then $\bar{g} = g$

ex) Inner product on $\mathcal{P}(\mathbb{R}^+ \cap 0)$ (weighted). Let $p, q \in \mathcal{P}(\mathbb{R}^+ \cap 0)$

$$\langle p, q \rangle = \int_{0}^{\infty} p(x)\,q(x)\,e^{-x}\,dx$$

We're ready to define an

Inner Product space: a vector space equipped with an inner product.