We want to recall the basics of the

CHANGE of COORDINATES

and we'll return to diagonalization of A afterwards

The coordinate transformation is effected by an isomorphism.

Suppose $\mathbb{B} \equiv \{b_i\}_{i=1}^n$ is a basis of the n-dimensional vector space V. Then, for any $v \in V$

$$v = x_1 b_1 + x_2 b_2 \dots x_n b_n = \sum_{i=1}^n x_i b_i$$

x_i are called the coordinates of the vector v in $\mathbb{B}$. We write these as

$$[v]_{\mathbb{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n .$$

so v is isomorphic to $[v]_{\mathbb{B}}$. It is an isomorphism between V and $\mathbb{F}^n$: it transforms the basis $\{b_i\}_{i=1}^n$ to the standard basis $\{\hat{e}_i\}_{i=1}^n$

here $\hat{e}_i = (0 \dots \underset{i^{\text{th}}\text{ entry}}{1} \dots 0)^T$, $\dim[\hat{e}_i] = n$.

Matrix (linear) transformation

$T: V \to W$

let $\mathbb{A} = \{a_i\}_{i=1}^{n}$ and $\mathbb{B} = \{b_i\}_{i=1}^{m}$

be the bases for V & W, respectively.

The matrix $T \in \mathbb{F}^{m \times n}$ relates the coordinates $[Tv]_{\mathbb{B}}$ and $[v]_{\mathbb{A}}$:

$$[Tv]_{\mathbb{B}} = [T]_{\mathbb{B}\mathbb{A}} [v]_{\mathbb{A}} .$$

matrix

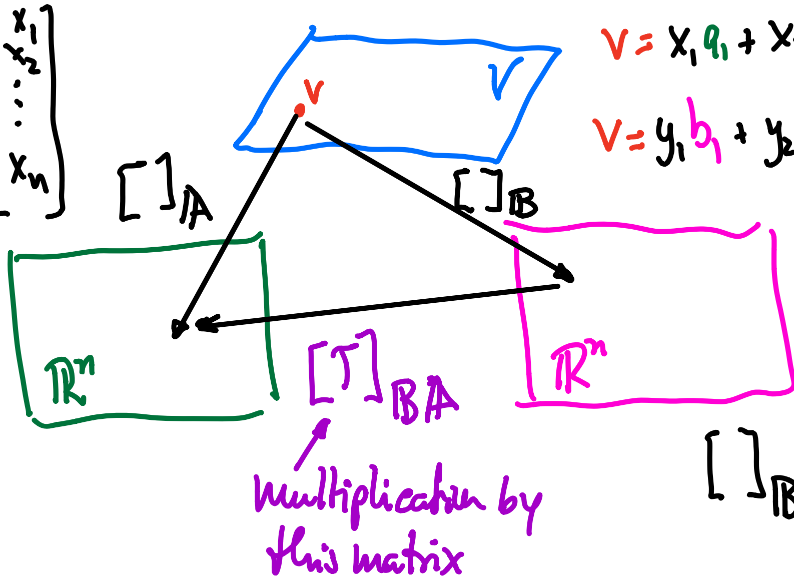
ex) S'pse $\mathbb{A} = \{a_i\}_{i=1}^{n}$ and $\mathbb{B} = \{b_i\}_{i=1}^{n}$

BOTH in V. The picture says it all:

$$[\ ]_{\mathbb{A}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$v = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$$v = y_1 b_1 + y_2 b_2 + \cdots + y_n b_n$$



$$[\ ]_{\mathbb{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

So $[T]_{\mathbb{B}/\mathbb{A}}$ relates the coordinates $[\ ]_{\mathbb{A}}$ and $[\ ]_{\mathbb{B}}$.

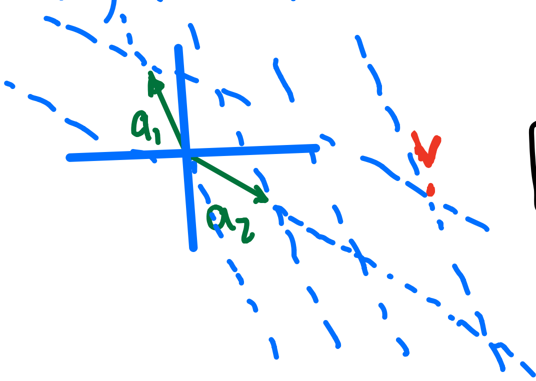
ex) Take $V=\mathbb{R}^n$. Let $\mathbb{A}=\{a_i\}_{i=1}^{n}$ and $\mathbb{B}=\{\hat{e}_i\}_{i=1}^{n}$

then $[a_i]_{\mathbb{B}} = a_i \quad i=1,2,\dots,n.$

and $[T]_{\mathbb{B}\mathbb{A}} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

a matrix consisting of columns made of a_i's. //

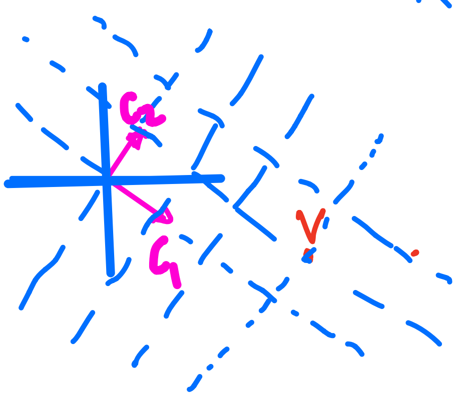
ex) Take $V=\mathbb{R}^2$



$V = x_1 a_1 + x_2 a_2 = 3a_1 + a_2$
(Given)

$[V]_{\mathbb{A}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\mathbb{A} = \{a_1, a_2\}$



$\mathbb{C} = \{c_1, c_2\}$

$$(*)\ \begin{cases} a_1 = 4c_1 + c_2 \\ a_2 = -6c_1 + c_2 \end{cases} \quad \text{(Given)}$$

Find $[V]_{\mathbb{C}}$

$$[v]_{\mathbb{A}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[v]_{\mathbb{C}} = [3a_1 + 1a_2]_{\mathbb{C}} = 3[a_1]_{\mathbb{C}} + 1[a_2]_{\mathbb{C}}$$

$$[v]_{\mathbb{C}} = \Big[[a_1]_{\mathbb{C}}, \ [a_2]_{\mathbb{C}} \Big] \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad (\exists)$$

Using ($\not\exists$)

$$[a_1]_{\mathbb{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad [a_2]_{\mathbb{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

Using ($\exists$)

$$[v]_{\mathbb{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

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ex) $V = \mathbb{R}^2$

$$b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \qquad \text{Let } \mathbb{B} = \{b_1, b_2\}$$

$$c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \quad c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \qquad \mathbb{C} = \{c_1, c_2\}$$

(A) Find the change of coordinate matrix from $\mathbb{C}$ to $\mathbb{B}$

$$[b_1]_{\mathbb{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad [b_2]_{\mathbb{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

i.e. $b_1 = c_1x_1 + c_2x_2$ & $b_2 = c_1y_1 + c_2y_2$

or

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1 \quad \& \quad \begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_2$$

Solve both systems for $\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix}$ simultaneously

The augmented system is

$$\left[\begin{array}{cc:cc} c_1 & c_2 & b_1 & b_2 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$[b_1]_{\mathbb{C}}$ $\qquad$ $[b_2]_{\mathbb{C}}$

So $$[T]_{\mathbb{C}\mathbb{B}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$\begin{matrix} x_1 \\ x_2 \end{matrix}$ $\qquad$ $\begin{matrix} y_1 \\ y_2 \end{matrix}$

(B) Find $$[T]_{\mathbb{B}\mathbb{C}} = \left[[T]_{\mathbb{C}\mathbb{B}}\right]^{-1} = \begin{bmatrix} -3/2 & -2 \\ 5/2 & 3 \end{bmatrix}$$

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Back to eigenvalues... We construct an $n \times n$ matrix as follows: V is a vectorspace with basis $\{b_i\}_{i=1}^n$. Choose $\mathbb{B} = \{b_i\}_{i=1}^n$ as the eigenvectors of A, presumed to have n LI eigenvectors. Then

$$[A]_{\mathbb{B}\mathbb{B}} = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

<u>Th'm</u>: A matrix $A \in \mathbb{F}^{n \times n}$ admits a representation $A = SDS^{-1}$, where D is a diagonal matrix and S is invertible iff $\exists$ a basis in $\mathbb{F}^n$ of eigenvectors of A. Further, D are e'values and S consists of columns of evectors of A.

Pf let $D = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_n)$ & $\{b_i\}_{i=1}^n = \mathbb{B}$, be columns of S.

Then $A = SDS^{-1}$ means $[A]_{\mathbb{B}\mathbb{B}} = D$ //

When is $A = SDS^{-1}$ possible?

If $A : V \to V$ and $\dim V = n$

$\mathrm{span}\, V = \{b_i\}_{i=1}^n$, b_i are distinct (LI) eigenvectors associated with $\{\lambda_i\}_{i=1}^n$ distinct evalues

Diagonalizing is a lot of work, and can be numerically challenging to do....

So why diagonalize?

ex) $x_{n+1} = M x_n$ $\quad M \in \mathbb{R}^{n \times n}$

$x_0 = X_0$ $\quad x_n \in \mathbb{R}^n \quad X_0 \in \mathbb{R}^n$

so $x_n = M^n X_0$ (by induction, easy to show)

$x_n = S D^n S^{-1} X_0$ (if M is diagonalizable)

ex) In economics $(I-B)^{\alpha} y = x \Rightarrow y = (I-B)^{-\alpha} x$

if $\|B\| \ll 1$ then $(I-B)^{-\alpha} \approx I + \alpha B$

$\therefore \quad y \approx (I + \alpha B) x$.

Generally, if $A = SDS^{-1}$

then $A^N = S D^N S^{-1}$

ex) $A^2 = (SDS^{-1})(SDS^{-1}) = S D^2 S^{-1}$

$A^3 = (SDS^{-1})(SDS^{-1})(SDS^{-1}) = S D^3 S^{-1}$, etc.