VECTOR SPACES

Read LADR 1.A, 1.B

Some basic definitions needed:

* Field (basic properties?)
* list (of length n)
* a set (of length n)

(how big can n be?, what's the difference between a set and a list?, when are 2 sets equivalent?)

Fundamental Concepts:

Vector, F^n (special)

Vector Space V

Skill to master:

How to prove a set of objects constitutes a vector space

Why Study Linear Algebra?

The basis for describing and understanding phenomena and constructs that obey LINEAR SUPERPOSITION

Linear superposition implies that the totality is equal to the sum of its parts. We'll be more precise later.

Linear Algebra gives us the tools to (a) show this, (b) find compact descriptions of the parts, (c) transform these parts into other equivalent parts, or show these equivalences.

Linear Algebra is, arguably, the mathematical framework of things we understand: the straight line and its generalizations.

Here is a complete list of applications considered in these pages:

- Abstract Thinking
- Chemistry
- Coding theory
- Coupled oscillations
- Cryptography
- Economics
- Elimination Theory
- Games
- Genetics
- Geometry
- Graph theory
- Heat distribution
- Image compression
- Linear Programming
- Markov Chains
- Networks
- Sociology
- The Fibonacci numbers
- Eigenfaces

I'll use the symbol $\underset{\sim}{f}$ (equivalent to $\vec{f}$ or $\mathbf{f}$ (bold) to indicate a vector

I'll use capital letters to denote a vector space

FINITE DIMENSIONAL VECTORS

A familiar example is the n-dimensional vector of "reals"

$$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \qquad u_j \in \mathbb{R}^1 \quad j = 1, 2 \ldots, n$$

"column" vector

$(u_1\, u_2 \ldots u_n)$ "row" vector

vector of "complex"

$$\underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad v_j \in \mathbb{C}^1 \quad j = 1, 2 \ldots, n.$$

$v_j = \alpha_j + i\beta_j\ ,\ \alpha_j \,\&\, \beta_j \in \mathbb{R}^1$

$i = \sqrt{-1}$

As in LADR we will use

$\mathbb{F}^n$ to represent either $\mathbb{R}^n$ or $\mathbb{C}^n$

(useful when something applies to either $\mathbb{R}^n$ or $\mathbb{C}^n$)

$\mathbb{R}^n$ is "a vector space"
$\mathbb{C}^n$ is a "vector space"
$\Big\}$ $\mathbb{F}^n$ is a vector space

Is $\underset{\sim}{u} \in \mathbb{R}^n$ also in $\mathbb{C}^n$?

Is $\underset{\sim}{u} \in \mathbb{C}^n$ also in $\mathbb{R}^n$?

if c is a scalar (a real or complex number), is it in $\mathbb{F}^1$?

if $\underset{\sim}{v} \in \mathbb{R}^m$, $m < n$, is it also in $\mathbb{R}^n$?

if $\underset{\sim}{v} \in \mathbb{R}^m$, $m > n$, is it also in $\mathbb{R}^n$?

... perhaps we should describe more precisely what a "vectorspace" is

Vector Space: A Collection of objects that form a vector space V

These objects obey 2 sets of properties: Addition (4)
Multiplication (4)

let $\underline{u}, \underline{v}, \underline{w} \in V$

Properties

Additive

1. Commutability $\underset{\sim}{u}+\underset{\sim}{v} = \underset{\sim}{v}+\underset{\sim}{u}$
2. Associativity $(\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w} = \underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w})$
3. The existence of additive inverse

   $\exists$ a vector $\underset{\sim}{w}$ s.t. $\underset{\sim}{v}+\underset{\sim}{w} = \underset{\sim}{\emptyset}$
4. The existence of $\underset{\sim}{0}$

   $\underset{\sim}{v}+\emptyset = \underset{\sim}{v}$

Multiplicative

def: $\alpha, \beta \in \mathbb{F}'$

1. $1\underset{\sim}{v} = \underset{\sim}{v}$ 1 is the scalar
2. $\alpha\beta\underset{\sim}{v} = \alpha(\beta\underset{\sim}{v})$
3. $\alpha(\underset{\sim}{u}+\underset{\sim}{v}) = \alpha\underset{\sim}{u}+\alpha\underset{\sim}{v}$
4. $(\alpha+\beta)\underset{\sim}{v} = \alpha\underset{\sim}{v}+\beta\underset{\sim}{v}$

The vectors so formed must live in V!

examples of vectors or "object"

$$\underset{\sim}{v} \in \mathbb{R}^n = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$ a "column" vector

$$(v_1\ v_2\ v_3 \ldots v_n)$$ a "row" vector

$$v_i \in \mathbb{R} \qquad i = 1, 2 \ldots, n$$

$$\underset{\sim}{w} \in \mathbb{R}^n \qquad \underset{\sim}{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$\underset{\sim}{u} + \underset{\sim}{w} = \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_n + w_n \end{pmatrix} \equiv \underset{\sim}{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

$$r_i \in \mathbb{R} \qquad \underset{\sim}{r} \in V$$

ex) $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^2$

$$\underset{\sim}{u} + \underset{\sim}{\emptyset} = \underset{\sim}{u} \qquad \underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underset{\sim}{u} + \underset{\sim}{v} = \emptyset$ yes if $\underset{\sim}{v} = -\underset{\sim}{u}$

$$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad -\underset{\sim}{u} = -\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}$$

is $1\underset{\sim}{u} = \underset{\sim}{u} = \begin{pmatrix} 1u_1 \\ 1u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underset{\sim}{u}$

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ex) $U = \{ f \mid f: \mathbb{R} \to \mathbb{R} \}$

$f_1 \in U \quad f_2 \in U$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$r f_1(x) = r f(x)$$

Ex) $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x+y+z = 0 \right\}$

subset of $\mathbb{R}^3$

$x, y, z \in \mathbb{R}^1$

show that P is a vector space

$$\begin{pmatrix} a \\ b \\ -(a+b) \end{pmatrix} + \begin{pmatrix} d \\ 0 \\ -d \end{pmatrix} = \begin{pmatrix} a+d \\ b \\ -(a+b)-d \end{pmatrix} \in P$$

$$r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix} \in P$$

We'll be using the symbol $\mathcal{P}_n$, to mean 2 different things, the context will make things clear

$$\mathcal{P}_3 = \{ \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \mid \alpha_i \in \mathbb{R} \}$$

at most degree 3.

$$\underset{\sim}{a} = \sum_{i=0}^{3} \alpha_i x^i \qquad \underset{\sim}{a} + \underset{\sim}{b} = \sum_{i=0}^{3} (\alpha_i + \beta_i) x^i$$

$$\underset{\sim}{b} = \sum_{i=0}^{3} \beta_i x^i$$

There's a sense in which we can think of $\mathcal{P}_3$ as equivalent to $\mathbb{R}^4$:

($\mathcal{P}_n$ as equivalent to $\mathbb{R}^n$)

via a "linear transformation":

$$\hat{e}_0 = x^0, \quad \hat{e}_1 = x^1, \quad \hat{e}_2 = x^2, \quad \hat{e}_3 = x^3$$

$$\underset{\frown}{u} = \alpha_0 \hat{e}_0 + \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3$$

Let's add 2 vectors:

$$\begin{array}{r} 1 - 2x + 0x^2 + 1x^3 \\ + \quad 2 + 3x + 7x^2 + 4x^3 \\ \hline 3 + x + 7x^2 + 5x^3 \end{array} \quad \text{OR:} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \\ 5 \end{pmatrix}$$

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$$P_3 = \{ \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \mid \alpha_i \in \mathbb{R}, i = 0, 1, 2 \\ \alpha_3 \neq 0 \}$$

Vector Space of polynomials of degree 3

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