HW2 Solution

(1) $Ax = b$

where $x = [x_1, x_2, x_3]^T$

$$A = \begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

(2) $\begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix} = A \quad \underset{\sim}{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$, put in augmented matrix

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{array}\right]$$

has a solution $\therefore \underset{\sim}{u} \in \text{span}[a_1, a_2]$

In fact $\underset{\sim}{u} = \frac{5}{2}\underset{\sim}{a_1} + \frac{3}{2}\underset{\sim}{a_2}$

(3) $$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & 3b_1 + \left(\frac{7}{2}\right)b_2 + b_3 \end{array}\right]$$

$Ax = b$ is consistent iff $3b_1 + \frac{7}{2}b_2 + b_3 = 0$

or $6b_1 + 7b_2 + 2b_3 = 0$

The set of such $\underset{\sim}{b}$ is a plane through the origin in $\mathbb{R}^3$

④ $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & -6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ 3 pivots ∴ spans $\mathbb{R}^3$

⑤ $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$

no free variable ∴ only trivial solution.

⑥ $\left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

⑦ The $A = [\underline{a}_1 \ \underline{a}_2]$ so $2\underline{a}_1 + 3\underline{a}_2 = \emptyset$

∴ $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ satisfies $A\underline{x} = \emptyset$

⑧ $\underline{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ needs to be satisfied by $A\underline{x} = \emptyset$

so $4\underline{a}_1 + 1\underline{a}_2 = \emptyset$. Hence $\underline{a}_2 = -4\underline{a}_1$

So choose any non-zero vector for first column of A & multiply that column by -4 to get $\underline{a}_2$ so, for example $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$

(5) Determine if the system has a nontrivial solution:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ -2x_1 - 3x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 9x_3 &= 0 \end{aligned}$$

(6) Describe all solutions of $A\underline{x} = \emptyset$, where A is row-equivalent to

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

(7) Given $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{bmatrix}$ find one non-trivial solution to $A\underline{x} = \emptyset$ by inspection

(8) Construct a 2×2 matrix A such that the solution of $A\underline{x} = \emptyset$ is the line through $(4,1)$ and the origin. Then find a vector $\underline{b}$ in $\mathbb{R}^2$ parallel to the solution set of $A\underline{x} = \underline{b}$ that is is not a line in $\mathbb{R}^2$ parallel to the solution set of $A\underline{x} = \emptyset$. Why does this not contradict

"The general solution to $A\underline{x} = \underline{b}$ is $\underline{x} = \underline{x}_h + \underline{x}_p$ where $A\underline{x}_p = \underline{b}$ and $A\underline{x}_h = \emptyset$"

(9) by inspection $\underset{\sim}{a}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\underset{\sim}{a}_2 = -2\underset{\sim}{a}_1 = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$

$\therefore$ linearly dependent

(10) $\underset{\sim}{v}_3 \in \text{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ iff $x_1\underset{\sim}{v}_1 + x_2\underset{\sim}{v}_2 = \underset{\sim}{v}_3$

has a solution.

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$$

$0 = 8$!

so $v_3 \notin \text{Span}(\underset{\sim}{v}_1, \underset{\sim}{v}_2)$

(11) for $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ to be LI then $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \emptyset$

in $x_1\underset{\sim}{v}_1 + x_2\underset{\sim}{v}_2 + x_3\underset{\sim}{v}_3 = \emptyset$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ -3 & 9 & -5 & 0 \\ -5 & 15 & h & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

for every h! x_2 is a free variable

So $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ are linearly dependent $\forall h$.

(12) True. If $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \underset{\sim}{v}_4\}$ are LI $\Rightarrow \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$

must also be LI.