Due: at the time of the in-class portion of the final. This work is voluntary. At most 20/100 points is given for each of the following problems. PLEASE submit neat work... points taken off for messy work.

① We considered the following system

$$(\mathbb{I} - C)\underset{\sim}{x} = \underset{\sim}{d}$$

The claim is that if $|C| < 1$ then

$$\lim_{N \to \infty} \sum_{i=0}^{N} C^i = (\mathbb{I} - C)^{-1} \quad (\ddagger)$$

(the norm of the matrix. Norms are ways of measuring the "size" of a matrix. We will not cover this in 341 (covered in 342))

Goal: to confirm ($\ddagger$) by taking $N = 0, 1, 2 \ldots$ and showing that the more terms you take in the finite sum, the closer you get to $(\mathbb{I} - C)^{-1}$

Use Matlab, python in the following

1. Create a matrix $C \in \mathbb{R}^{50\times 50}$ and $A \in \mathbb{R}^{50\times 50}$ where $A = (\mathbb{I}-C)^{-1}$

```
n=50;
C=zeros(n) ; A=zeros(n);
for ii=1:n
    dx=10^-5+(0.99-10^-5)*rand;
    C(ii,ii)=dx;
    A(ii,ii)=1/(1-dx);
end
```

C is a diagonal matrix, and $A = (\mathbb{I}-C)^{-1}$ also diagonal.

2. Confirm that $A\cdot(\mathbb{I}-C) = \mathbb{I}$.

3. Let $M_N = \sum_{i=0}^{N} C^i$. Let $W = \mathbb{I} - C$

The eigenvalues of C are the diagonal entries and if $|\lambda_i|<1$ for $1\le i\le n$, then $(\ddagger)$ holds:

The claim is thus that

$$\lim_{N\to\infty} M_N * W = \mathbb{I}.$$

Make a table and/or figure that consists of

N	norm(M_N*W,2) = D
0	⋮
1	⋮
2	⋮
3	⋮
4	⋮
⋮	⋮

or plot(N, D).

(4) D should approach 1 as $N \to \infty$. Confirm this with table or plot.

II In class we often said that in solving

$$(*) \quad \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b} \qquad A \in \mathbb{R}^{n\times n};\ \underset{\sim}{x}, \underset{\sim}{b} \in \mathbb{R}^n$$

for the unknown $\underset{\sim}{x}$, we avoid direct use of the formula

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b} \quad \text{(when } n \text{ is large)}$$

because computing A^{-1} is very expensive (and sometimes very numerically unstable).

There are a whole class of methods for solving (approximately) (‡) called

Iterative Methods

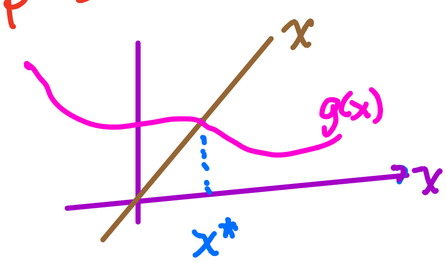
Finding the root(s) of an equation

$$h(x) = 0$$

can be written as a fixed point problem

$$h(x) = x - g(x) = 0$$

Hence the root(s) of $h(x) = 0$ satisfy $x = g(x)$ a fixed point problem



x^* is a fixed point of $x = g(x)$

The fixed point is unique over an interval in x for which $|g'(x)| < 1$. Assume $|g'(x)| < 1$ then

the map $\quad x_{n+1} = g(x_n) \quad n = 0, 1, 2, \ldots$

x_0 an initial guess

As $n \to \infty$ $\; x_n \to x^*$. That is, will converge to $x^* = g(x^*)$.

Let's go back to solving $A x = b$.

We'll write $A = T - D$ (many ways to do this)

$$T\underset{\sim}{x} - D\underset{\sim}{x} = b$$

or $$T\underset{\sim}{x} = D\underset{\sim}{x} + b$$

The idea is to choose T to be easily invertible, so

$$\underset{\sim}{x} = T^{-1}(D\underset{\sim}{x} + \underset{\sim}{b}) = T^{-1}D\underset{\sim}{x} + T^{-1}\underset{\sim}{b}$$

Now, in order to exploit the fixed point idea, choose T so that $|T^{-1}D| < 1$

so that $\begin{cases} \underset{\sim}{x}_{n+1} = T^{-1}D\,\underset{\sim}{x}_n + T^{-1}b, & n = 0, 1, 2 \ldots \\ \underset{\sim}{x}_0 \text{ a guess} \end{cases}$

will converge as $n \to \infty$ to the solution $\underset{\sim}{x}$, which is a solution to $A\underset{\sim}{x} = \underset{\sim}{b}$.

Jacobi Iteration: suppose that the matrix A is "diagonally dominant": this is the case when $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ $1 \leq i \leq n$

the absolute value of the diagonal entry > the sum of the absolute value of the off-diagonal elements in each row.

In that case, let $T = \text{diag}(A)$

A matrix of the diagonal elements of A

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \\ a_{21} & a_{22} & & \\ & & \ddots & a_{nn} \end{pmatrix} = T - D$$

$$T = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & a_{33} & \\ 0 & & & \ddots a_{nn} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & \ddots & & \\ \vdots & & & \ddots & \\ -a_{n1} & & & & 0 \end{pmatrix}$$

So $\underset{\sim}{x}_{n+1} = T^{-1} D \underline{x}_n + T^{-1} \underset{\frown}{b}$

$\underset{\frown}{x}_0$ some initial guess.

In matlab

```
n=100;
T=zeros(n); D=T;
A=rand(n)+n*eye(n);
s=rand(n,1);
b=A s ;
```

So for $A\underset{\sim}{s}=\underset{\sim}{b}$, we know what the solution $\underset{\sim}{s}$ is.

A has been built so that it is diagonally dominant, but check yourself.

Assign T the matrix with diagonal entries of A. Then $D=T-A$. Find T^{-1} (analytically)

Pick so random $\underset{\sim}{x}_0 \in \mathbb{R}^n$, write a loop

```
mx=50;
x(:,1)=rand(n,1);
K=T^-1; % Finding the inverse of T is
        % trivial. Do it analytically.

L=K*D;
```

```
B = K*b;

for jj = 1:mx

  x(:,jj+1) = L*x(:,jj) + B;
  normdist(jj) = norm(x(:,jj+1) - S(:,1));

end
plot(normdist)
```

From the plot, what can you conclude?