We will denote the standard basis for the n vectors of $\mathbb{R}^n$, each of size n, by $\hat{e}_i$ $i=1,2,\dots,n$.

TRANSFORMATION EXAMPLES (the standard basis is helpful)

ex) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ linear.

Let $\underset{\sim}{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ & $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \underset{\sim}{m}$

and $y_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $y_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

(a) Write $\underset{\sim}{x}$ as a linear combination of the standard basis,

$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: Let z_1, z_2 be numbers.

$\underset{\sim}{x} = z_1 \hat{e}_1 + z_2 \hat{e}_2 = 5\hat{e}_1 - 3\hat{e}_2$ (by inspection)

(b) Find $T(\underset{\sim}{m}) = m_1 \hat{e}_1 + m_2 \hat{e}_2$.

(c) Find $T(\underset{\sim}{x}) = T(5\hat{e}_1) + T(-3\hat{e}_2)$.

$\therefore \quad T(\underset{\sim}{x}) = 5T\hat{e}_1 - 3T\hat{e}_2.$

(d) know that $T\hat{e}_1 = \underset{\sim}{y}_1$ & $T\hat{e}_2 = \underset{\sim}{y}_2$,

Find $T(\underset{\sim}{x}) = 5T\hat{e}_1 - 3T\hat{e}_2$:

$$T(\underset{\sim}{x}) = 5\underset{\sim}{y}_1 - 3\underset{\sim}{y}_2 = 5\begin{pmatrix}2\\5\end{pmatrix} - 3\begin{pmatrix}-1\\6\end{pmatrix}$$

$$= \begin{pmatrix}13\\7\end{pmatrix}.$$

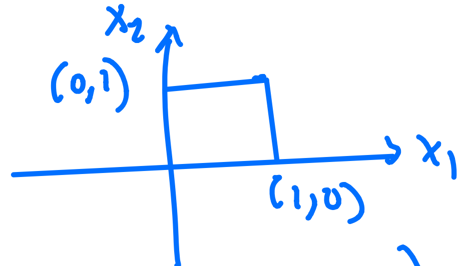
if $T(x) = x_1\underset{\sim}{y}_1 + x_2\underset{\sim}{y}_2 = \begin{pmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{pmatrix}$

$$\begin{pmatrix}2 & -1\\5 & 6\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}13\\7\end{pmatrix} \Rightarrow \begin{matrix}x_1 = 5\\x_2 = -3\end{matrix}$$

//

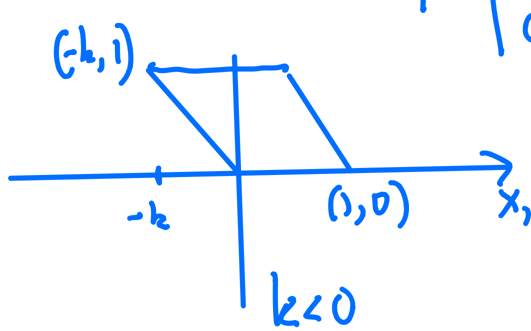
MORE ELEMENTARY TRANSFORMATIONS:

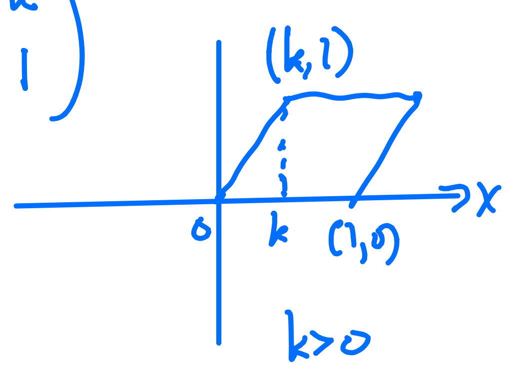
SHEAR:



Horizontal Shear

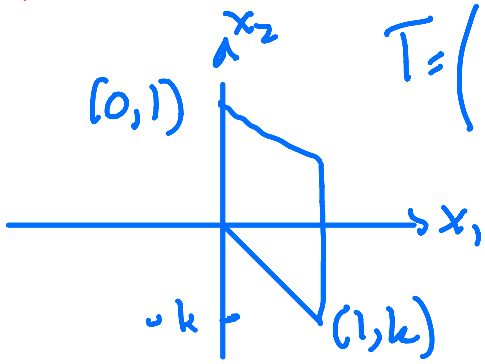
$$T = \begin{pmatrix}1 & k\\0 & 1\end{pmatrix}$$



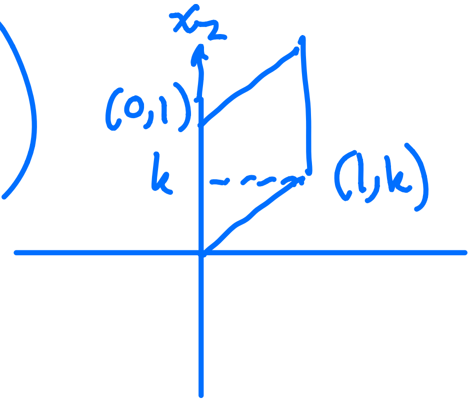


Vertical Shear

$$T = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$



$k < 0$



$k > 0$ //

ex) Find the matrix T such that

$T : \mathbb{R}^3 \to \mathbb{R}^2$ and

$$T\hat{e}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad T\hat{e}_2 = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \qquad T\hat{e}_3 = \begin{pmatrix} 3 \\ -8 \end{pmatrix}.$$

Here, $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \in \mathbb{R}^{2\times 3}$, let $\underset{\sim}{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

then $T\underset{\sim}{x} = Tx_1\hat{e}_1 + Tx_2\hat{e}_2 + Tx_3\hat{e}_3$;

$$Tx_1\hat{e}_1 = x_1 T\hat{e}_1 = x_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$Tx_2\hat{e}_2 = x_2 T\hat{e}_2 = x_2\begin{pmatrix} -2 \\ 9 \end{pmatrix} \quad \& \quad Tx_3\hat{e}_3 = x_3 T\hat{e}_3 = \begin{pmatrix} 3 \\ -8 \end{pmatrix}.$$

$\therefore\ T = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$ //

ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ vertical shear transformation that maps $\hat{e}_1$ into $\hat{e}_1 - 3\hat{e}_2$ but leaves $\hat{e}_2$ unchanged:

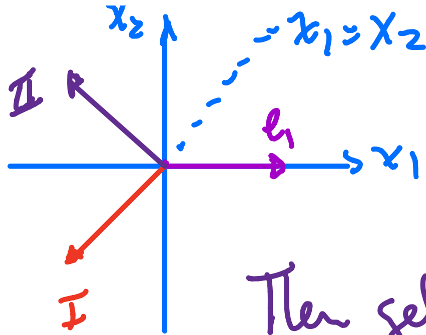
$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T\hat{e}_2 = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\hat{e}_1 = \hat{e}_1 - 3\hat{e}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$ //

ex) T rotate points (clockwise) by $-3\pi/4$, then reflects points about horizontal x_1 axis

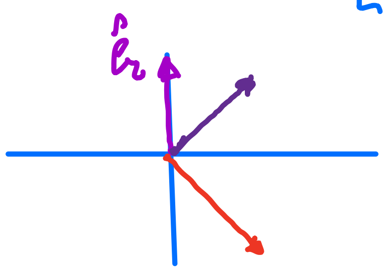


I: $R\hat{e}_1 = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$
Rotation

Then gets reflected: so

$$T\hat{e}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

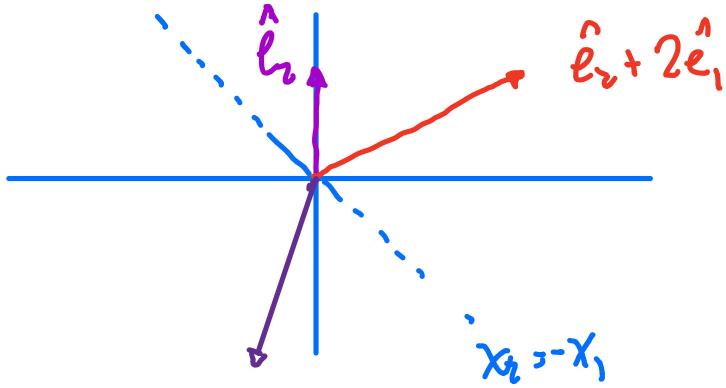
$$A = \begin{bmatrix} T\hat{e}_1 & T\hat{e}_2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & b \\ 1/\sqrt{2} & d \end{bmatrix}.$$ Now work out $\begin{bmatrix} b \\ d \end{bmatrix}$:



$T\hat{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So

$A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is the final expression for the transformation. //

ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ first horizontal shear that transforms $\hat{e}_2 \to \hat{e}_2 + 2\hat{e}_1$, leaving $\hat{e}_1$ unchanged, and then reflects points through the line $x_2 = -x_1$.



I: $T = [a_1 \, a_2]$

$$T\hat{e}_2 = \begin{pmatrix}0\\1\end{pmatrix} + 2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$0a_1 + 1a_2 = \begin{pmatrix}2\\1\end{pmatrix}$$

$$a_2 = \begin{pmatrix}2\\1\end{pmatrix}$$

$$T\hat{e}_1 = \hat{e}_1 \Rightarrow a_1 = \begin{pmatrix}1\\0\end{pmatrix}$$

No reflection $T = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$. Now work out the reflection:

Reflection about $x_2 = -x_1$: $\hat{e}_1 \rightarrow -\hat{e}_2$ $\quad \hat{e}_2 \rightarrow -\hat{e}_1$

$$\therefore \quad T = \begin{bmatrix}0 & -1\\-1 & -2\end{bmatrix}$$ final