Linear Transformations, continued:

Some linear transformations.

Ex) Given: $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$

$\underset{\sim}{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$

(a) Find the "image" of $\underset{\sim}{u}$ under A:

$$A\underset{\sim}{u} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix} \in \mathbb{R}^3$$

(b) Find $T(\underset{\sim}{x})$, where $T = A$

$$A\underset{\sim}{x} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

(c) Find $\underset{\sim}{v} \in \mathbb{R}^2$ whose image under $T = A$ is $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$A\underset{\sim}{v} = \underset{\sim}{b}$ find $\underset{\sim}{v}$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$A \quad b$

$$\underset{\sim}{v} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$$

(d) Is there more than 1 $\underset{\sim}{v}$ whose image under $T = A$ is b?

$\underset{\sim}{v}$ is unique (see c) //

LINEAR TRANSFORMATION

$T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if

$$\left.\begin{array}{ll} (*) & T(\underline{u}+\underline{v}) = T\underline{u} + T\underline{v} \\ & \text{and} \\ (**) & T(c\underline{u}) = cT\underline{u} \end{array}\right\} \forall\, u, v \text{ in domain of } T.$$

(*) linear transformations preserve vector addition.
(linear superposition)

(**) " " scalar multiplication

The above justifies linear superposition & the process of seeking elementary vectors (fundamental degrees of freedom) with which to represent or analyze the action of a map. //

ex) A nonlinear transformation:

$$y = x^2$$

$$T: \mathbb{R} \to \mathbb{R} \qquad T: x \mapsto x^2$$

let $a, b \in \mathbb{R}$

$$\left.\begin{array}{l} T(a+b) = (a+b)^2 = a^2 + 2ab + b^2 \\ T(a) = a^2 \qquad T(b) = b^2 \end{array}\right\} T(a+b) \neq T(a) + T(b)$$

//

An AFFINE TRANSFORMATION: $T: \mathbb{R}^m \to \mathbb{R}^n$ has the form $T(\underline{x}) = A\underline{x} + \underline{b}$

$A \in \mathbb{R}^{m \times n}$ $\underline{b} \in \mathbb{R}^n$, constant vector

T is not a linear transformation (unless $\underline{b} = \emptyset$)

$$T(a\underline{x}) = aA\underline{x} + b$$

c is a number. Does not preserve multiplication by a scalar, for example

$$T(\emptyset) = A\emptyset + \underline{b} \neq \emptyset \, !$$

The identity matrix: $\mathbb{I}_n = \delta_{ij}$ is the $n \times n$ matrix with only 1's on the diagonal

$$\mathbb{I}_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

we might omit the subscript n, if the context is clear: $\mathbb{I} = \mathbb{I}_n$

δ_{ij} is the Kronecker Delta function

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$i, j = 1, 2, \ldots, n$, are rows & columns, respectively.

Linear Transformations in $\mathbb{R}^2$

<u>DILATION</u> $\quad A = c\,\mathbb{I}$

c is a number, $c \geq 0$

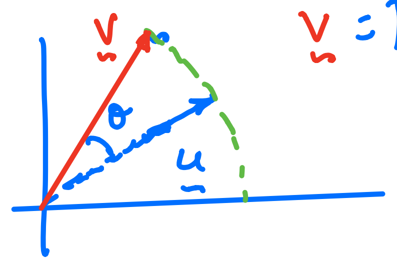
$$A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

$x \in \mathbb{R}^2 \qquad A\underset{\sim}{x} = c\underset{\sim}{x}$

Rotations

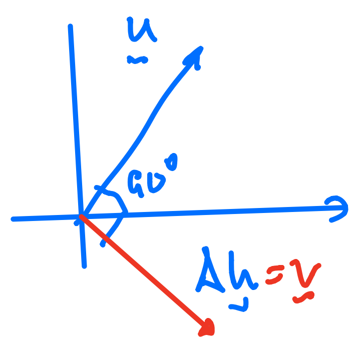
$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = T(\theta)$$

$\underset{\sim}{v} = T(\theta)\underset{\sim}{u}$



$T(\theta)$ rotation by θ, preserves length.

ex) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



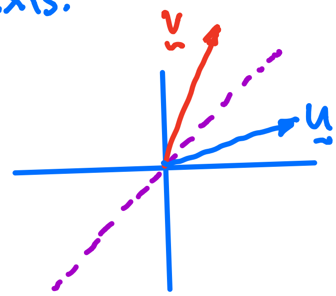
$$A\underset{\sim}{u} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underset{\sim}{v}.$$

$\underset{\sim}{v} = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$ Note that $|\underset{\sim}{v}| = |\underset{\sim}{u}|$.

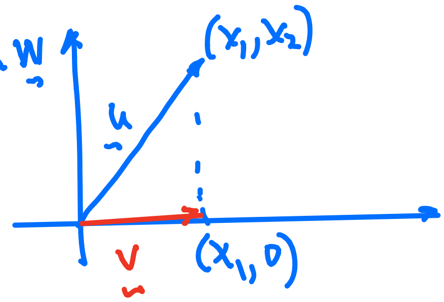
Reflection: for example, reflect about x axis, or y axis, or some other axis.

ex) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflects about $x=y$

$\underset{\sim}{v} = A\underset{\sim}{u}$



Projection: projects in the direction $\underset{\sim}{w}$



ex) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \underset{\sim}{v}$ the direction $\underset{\sim}{w}$ is aligned with x axis.

Def: The Standard Basis in $\mathbb{R}^n$ is the set of vectors

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{n} \Bigg\} n$$