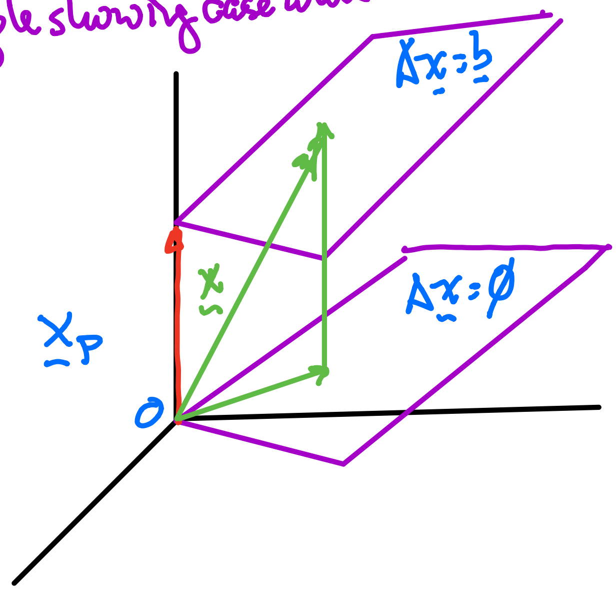
Thm: Let $A\underset{\sim}{x}=\underset{\sim}{b}$ consistent. Let $\underset{\sim}{x}_p$ be a solution of $A\underset{\sim}{x}=\underset{\sim}{b}$ $\Rightarrow$ The general solution of $A\underset{\sim}{x}=\underset{\sim}{b}$ is $\underset{\sim}{x}=\underset{\sim}{x}_h+\underset{\sim}{x}_p$

where $\underset{\sim}{x}_h$ satisfies $A\underset{\sim}{x}_h=\emptyset$, $A\underset{\sim}{x}_p=\underset{\sim}{b}$. //

$\underset{\sim}{x}_h$ may be the trivial solution.

Example showing case where there are 2 free parameters in $\mathbb{R}^3$:



The solutions represent two parallel sets
$\underset{\sim}{x}=\underset{\sim}{x}_h+\underset{\sim}{x}_p$

Solution is a translation of $\underset{\sim}{x}_h$

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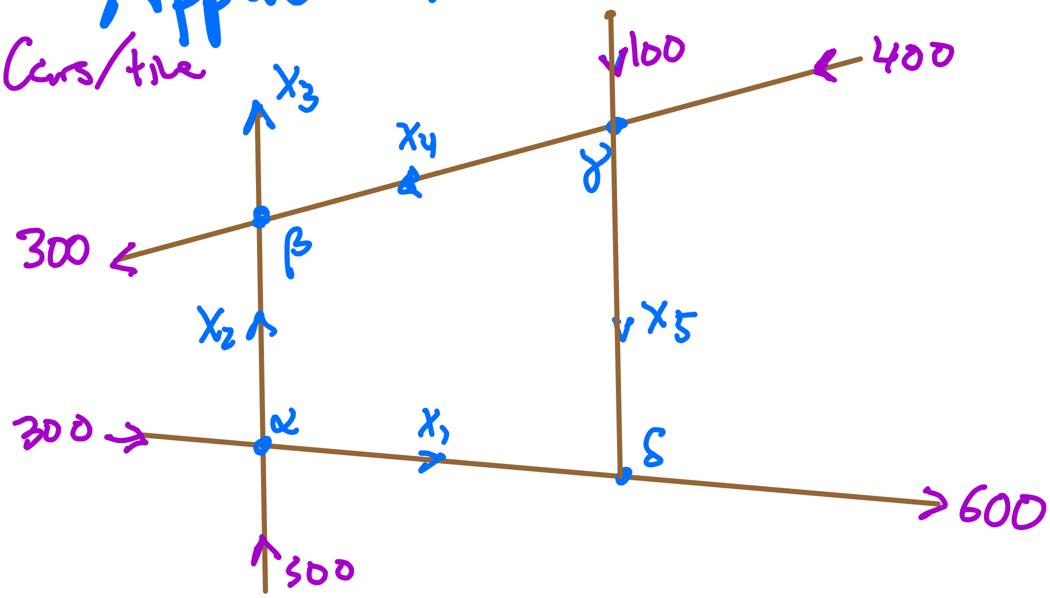
Application: network flow

Cars/time



Intersection	Flow In	Flow Out
α	$300+500$	x_1+x_2
β	x_2+x_4	x_3+300
γ	$100+400$	x_4+x_5
δ	x_1+x_5	600

$(\heartsuit)$ TOTAL INFLOW = TOTAL OUTFLOW (No cars are parked or add/removed)

$$500+300+100+400$$

$$= 300+x_3+600$$

$$\Rightarrow x_3 = 400$$

$(\heartsuit)$ is a "Conservation law" and determines at least one parameter.

The Resulting system:

$$\begin{aligned}
x_1 + x_2 &= 800\\
x_2 - x_3 + x_4 &= 300\\
x_4 + x_5 &= 500\\
x_1 + x_5 &= 600\\
x_3 &= 400
\end{aligned}$$

Let $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.

Row Reduce

1. $x_1 = 600 - x_5$
2. $x_2 = 200 + x_5$
3. $x_3 = 400$
4. $x_4 = 500 - x_5$
5. $x_5 =$ free

N.B. None of the x_i's can be negative (since 1 way streets)

$\boxed{③\ x_3 = 400}$ ✓

Implies that $x_5 \geq 0$ (5)

$\boxed{500 \geq x_5 \geq 0}$ by (4)

(1) $x_1 = 600 - x_5$ ✓ $\boxed{100 \leq x_1 \leq 600}$

$\boxed{200 \leq x_2 \leq 700}$ $\boxed{0 \leq x_4 \leq 500}$

//

LINEAR INDEPENCE

Let $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\} \in \mathbb{R}^n$ are said to be linearly independent iff the equation

$$(*) \quad x_1 v_1 + x_2 v_2 + \cdots x_p v_p = \emptyset$$

can only be satisfied with $\boxed{x_i = 0}$

x_i's are numbers.

//

Suppose $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix}$ then $\{\mathbf{v}_i\}_{i=1}^{p}$ are LI (linearly independent) iff

the solution of $A\mathbf{x} = \mathbf{0}$ $\quad$ (**)

is $\mathbf{x} = \mathbf{0}$ the trivial solution. //

So (*) & (**) are equivalent

$\therefore$ Can use row reduction strategy to determine whether a set of vectors, all living in $\mathbb{R}^n$ are LI