The solutions of $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$

Thm $A\underline{x} = \underline{b}$ has a solution iff $\underset{\sim}{b}$ is a linear combination of the columns of A, i.e. $\underset{\sim}{b} \in \text{span}\{a_i\}_{i=1}^{n}$

$$\underline{b} = A\underline{x} = x_1 a_1 + x_2 a_2 \cdots + x_n a_n$$

it might have more than 1 solution //

$A \in \mathbb{R}^{m \times n}$, is composed of (vectors) columns a_i

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$a_i \in \mathbb{R}^m \quad b \in \mathbb{R}^m \quad \underset{\sim}{x} \in \mathbb{R}^n$

ex) $\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \end{array}$ $\qquad \underset{\sim}{b} = A\underset{\sim}{x}$

row ops

want $(b_1, b_2, b_3) = \underset{\sim}{b} \in \text{span}(a_1, a_2, a_3)$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & \underbrace{b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_3)}_{=\,0 \text{ required}} \end{array}\right]$$

for consistency

$$\left.\begin{aligned} 14x_2 + 10x_3 &= b_2 + 4b_1 \\ x_1 + 3x_2 + 4x_3 &= b_1 \end{aligned}\right\} \text{allow you to choose either } x_i, b_i$$

Thm The following are ALL true or ALL FALSE:

Let $A \in \mathbb{R}^{m \times n}$

(a) $b \in \mathbb{R}^m$ $\quad A\underset{\sim}{x} = \underset{\sim}{b}$ has a solution

(b) $b \in \mathbb{R}^m$ $\quad$ is span cols of A

(c) The columns of A span $\mathbb{R}^m$

(d) A has a pivot in every row
(not augmented matrix)

Some important matrix properties

Given $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^m$

$A \in \mathbb{R}^{m \times n}$

$$A(\underset{\sim}{u} \pm \underset{\sim}{v}) = A\underset{\sim}{u} \pm A\underset{\sim}{v}$$

Also

$$A(c\underset{\sim}{u}) = c\,A\underset{\sim}{u} \quad c \text{ a constant.}$$

Important case: $A\underline{x} = \emptyset$

"Null Space Problem"

or "Homogeneous Problem"

Important case: $A\underline{x} = \emptyset$

"Null Space Problem"

or "Homogeneous Problem"

$A \in \mathbb{R}^{m \times n}$, is composed of (vectors) columns a_i

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$a_i \in \mathbb{R}^m \quad b \in \mathbb{R}^m \quad \underset{\sim}{x} \in \mathbb{R}^n$

Solutions of the Null Space Problem
also called the "Homogeneous" Problem.

The general solution to $A\underset{\sim}{x} = \underset{\sim}{b}$

$\underset{\sim}{b} \neq \emptyset$

is $\underset{\sim}{x} = \underset{\sim}{x}_h + \underset{\sim}{x}_p$

where $\begin{cases} A\underset{\sim}{x}_h = \emptyset \\ A\underset{\sim}{x}_p = \underset{\sim}{b} \neq \emptyset \end{cases}$

Note: $A(\underset{\sim}{x}_h + \underset{\sim}{x}_p) = A x_h + A x_p = \underset{\sim}{b}$

$\Rightarrow$ Need to understand $A\underset{\sim}{x}_h = \emptyset$

$ax = b$ a, b #'s.

$x = 0 + \frac{b}{a}$

is also true in 1D

① $A\underline{x} = \emptyset$

always has $\overbrace{\underset{\sim}{x} = \emptyset}^{\text{Trivial solution}}$ as a solution.

But can have $\underset{\sim}{x}_n \neq \emptyset$ as well!

$$\left[\begin{array}{c|c} A & 0 \end{array}\right] \sim \left[\begin{array}{c|c} \boxed{\cdots} & 0 \\ 0 & \end{array}\right]$$

② $A\underset{\sim}{x} = 0$ has a non-trivial solution iff the equations have at LEAST one free variable.

ex) Determine if $A\underset{\sim}{x} = \emptyset$ has a non-trivial solution:

$$[A|\emptyset] = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \underline{\underline{3}} & 5 & -4 & 0 \\ 0 & \underline{\underline{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

x_3 is free

$x_1 = \frac{4}{3} x_3$

$x_2 = 0$

$$\underset{\sim}{x} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} x_3$$

nontrivial solution.

x_3 free

$\underset{\sim}{x}$ "Lives" in the space spanned by

$$\underset{\sim}{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

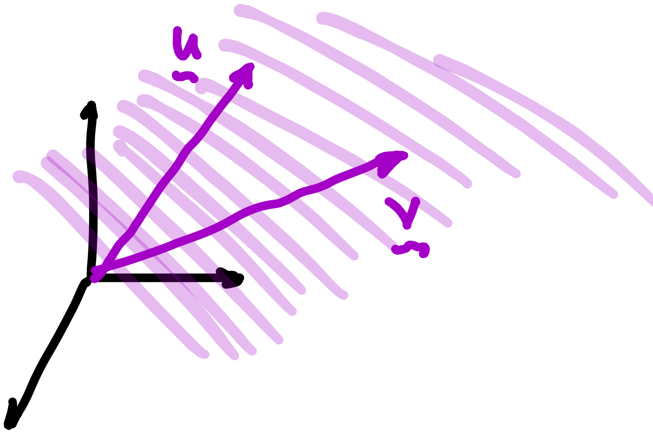
ex) Describe all solutions to $A\underset{\sim}{x} = \emptyset$, $\underset{\sim}{x} \in \mathbb{R}^3$

$10x_1 - 3x_2 - 2x_3 = 0$ $\quad$ $x_1 = 0.3x_2 + 0.2x_3$

let x_2, x_3 be free

$$\underset{\sim}{x} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{u}} + x_3 \underbrace{\begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}}_{\underset{\sim}{v}}$$

note: $\underset{\sim}{u} \neq c\,\underset{\sim}{v}$



plane spanned by $\underset{\sim}{u}$ & $\underset{\sim}{v}$