Vector Equations

Vector: is a member of a "vector space"

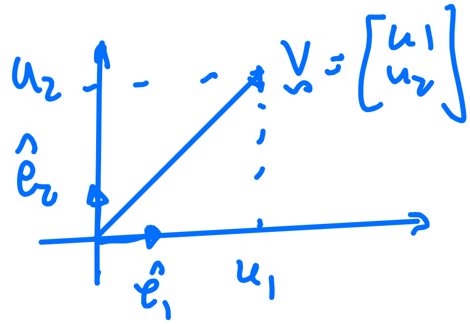
Finite Dimensional Vectors (over reals)

The dimension is $n \geq 0$, integer

$\underset{\sim}{v} \in \mathbb{R}^n$ (matrix 1 column or 1 row)

$\underset{\sim}{v} = \emptyset$ "zero vector"

ex) $\underset{\sim}{v} \in \mathbb{R}^2$ $\quad \underset{\sim}{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\quad u_i$ are real #'s.



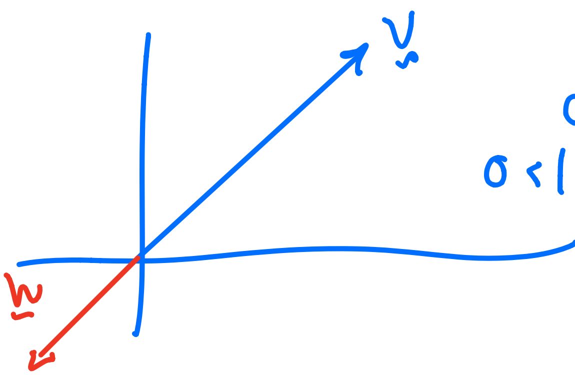
$$\underset{\sim}{v} = u_1 \hat{e}_1 + u_2 \hat{e}_2$$

$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard vectors

You can add / subtract $\begin{bmatrix} a \\ b \end{bmatrix} \pm \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \pm c \\ b \pm d \end{bmatrix}$

Multiply by a constant $\underset{\sim}{w} = c\,\underset{\sim}{v}$

ex)



$c < 0$

$0 < |c| < 1$

Find the magnitude of a vector in $\mathbb{R}^n$

$$|\underset{\sim}{v}| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} \geq 0$$

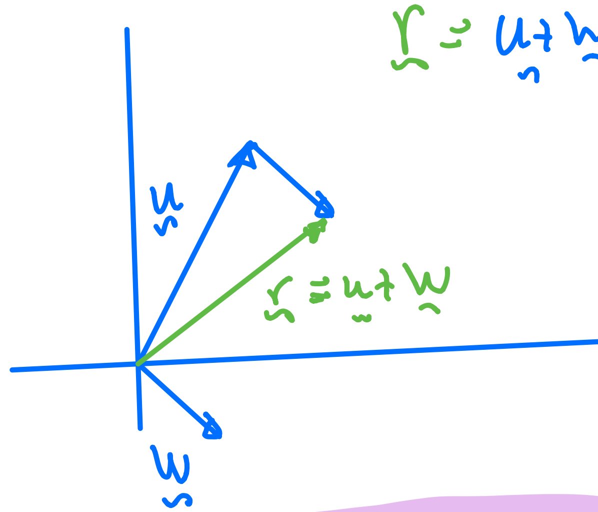
where $\underset{\sim}{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

$|\emptyset| = 0$

Geometric Interpretation of the addition of 2 vectors

$\underset{\sim}{r} = \underset{\sim}{u} + \underset{\sim}{w}$

in $\mathbb{R}^2$



Can add 2 or more vectors of same dimension.

<u>Algebra of $\mathbb{R}^n$</u>

* The usual axioms for additivity apply.
* Multiplication by a constant.
  if $u \in \mathbb{R}^n$ then $cu \in \mathbb{R}^n$
  c constant.

Members of $\mathbb{R}^n$ are "closed" under addition & multiplication by a scalar.

if $\underset{\sim}{u} \in \mathbb{R}^n$ $\exists$ $\underset{\sim}{v} \in \mathbb{R}^n$ s.t.

$\underset{\sim}{u} + \underset{\sim}{v} = \emptyset$ additive inverse exists

and $1 \underset{\sim}{u} = \underset{\sim}{u}$

$0 \underset{\sim}{u} = \emptyset$ //

Linear Combinations (key concept)

Given $\underset{-}{v}_1, \underset{-}{v}_2, \ldots, \underset{-}{v}_p \in \mathbb{R}^n$

$\underset{\sim}{y} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots c_p \underset{\sim}{v}_p$, c_i are numbers

$\underset{\sim}{y}$ is a linear combination

$\underset{\sim}{y} \in \mathbb{R}^n$ //

Linear Combination of column vectors: $\underset{\sim}{b} = A \underset{\sim}{x}$ $(\star)$

where $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_p \\ | & | & & | \end{bmatrix}$. $x \in \mathbb{R}^p$

$\star$ $\underset{\sim}{b} = a_1 x_1 + a_2 x_2 \cdots + a_p x_p$

so $\underset{\sim}{b}$ is a linear combination //

ex) Let $\underset{\sim}{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$ $\quad \underset{\sim}{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

$\underset{\sim}{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

Determine whether $\underset{\sim}{b}$ can be generated as a linear combination of $\underset{\sim}{a}_1$ & $\underset{\sim}{a}_2$

i.e. show that $\quad c_1 \underset{\sim}{a}_1 + c_2 \underset{\sim}{a}_2 = \underset{\sim}{b}$

need to find c_1 & c_2 :

$$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underset{\sim}{b}$$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$c_1 + 2c_2 = 7$

$-2c_1 + 5c_2 = 4$

$-3c_1 + 6c_2 = -3$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -3 & 6 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

$\underset{\sim}{b} = 3\underset{\sim}{a}_1 + 2\underset{\sim}{a}_2$ $\quad$ Yes //

$\therefore$ A vector eq (a linear combination)
has the same solution as the augmented
matrix problem

Span: If $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p \in \mathbb{R}^n \Rightarrow$ set of
all linear combinations of $\{\underline{v}_i\}_{i=1}^{p}$
is $\text{Span}[\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p]$.
It is also a subset of $\mathbb{R}^n$
(spanned, or generated by) $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p$. //

ex) Given $\underline{b} \in \mathbb{R}^n$

and $\underline{v}_i \in \mathbb{R}^n$ $\quad i = 1, 2, \dots, p$

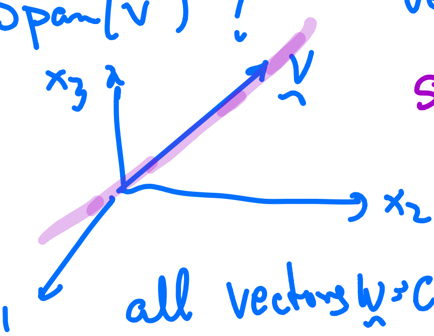
Is $\underline{b}$ in the span of $\{\underline{v}_i\}_{i=1}^{p}$?

Yes if $\underline{b}$ is a linear combination
of a set (or subset) of $\{v_i\}_{i=1}^{p}$. //

Geometric Description of Span{$\underset{\sim}{v}$} & Span($\underset{\sim}{u}, \underset{\sim}{v}$) in $\mathbb{R}^3$:

what is the geometry of

Span(v) ? $v \in \mathbb{R}^3$

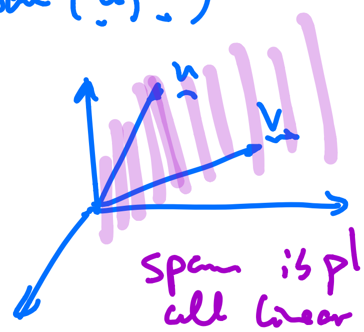


span($\underset{\sim}{v}$) is the collection of vectors formed by $c\underset{\sim}{v}$, c, constant.

all vectors $\underset{\sim}{w} = c\underset{\sim}{v}$, c is a constant.

Span ($\underset{\sim}{u}, \underset{\sim}{v}$) $u, v \in \mathbb{R}^3$

$\underset{\sim}{u} \neq c\underset{\sim}{v}$



span is plane containing all linear combo's of $\underset{\sim}{u}, \underset{\sim}{v}$.

ex) let $\underset{\sim}{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\underset{\sim}{a}_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix}$

Is $\underset{\sim}{b}$ in the plane spanned by $\underset{\sim}{a}_1$ & $\underset{\sim}{a}_2$?

if so $\underset{\sim}{b} = x_1\underset{\sim}{a}_1 + x_2\underset{\sim}{a}_2$

hence, $\underset{\sim}{b} = A\underset{\sim}{x}$, has a solution

$$\left[\begin{array}{cc|c} 1 & 3 & -3 \\ 2 & 4 & 8 \\ 3 & 5 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

inconsistent

$\Rightarrow$ $\underset{\sim}{b}$ is $\underline{\text{not}}$ in the span$(\underset{\sim}{a_1}, \underset{\sim}{a_2})$

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