Theorem Each matrix is row-equivalent to 1 & only 1 reduced echelon matrix

Pf: Use contradiction:

Given A $\rightarrow$ assume B & C are equivalent and show that $B - C = \emptyset$ to avoid contradictions. //

def: Row Equivalent B & C are row equivalent matrices if C is produced from B by elementary row operations.

Pivot Position: lead 1 in the reduced echelon form of the equivalent matrix. A pivot column is a column that contains a pivot. //

ex) $A = \begin{array}{c} \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \end{array} \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}$ Find row equivalent matrix:

i.e. Put in echelon form using 3 elementary row ops.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \qquad E_1 \leftrightarrow E_4$$

$$\begin{bmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \qquad (//) \text{ are pivots}$$

$$\begin{bmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ 0 & \underline{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

then swap E_3 & E_4

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Leftarrow B\underset{\sim}{x} = \underline{d}$$

pivots are in position 1, 2, 4. (1, 2, 4 are "pivot columns")

Suppose $A = \left(B \,\middle|\, d \right)$

$\underset{\sim}{x} = \mathbb{R}^4 \qquad \underline{d} \in \mathbb{R}^4$

$B \in \mathbb{R}^{4\times 4}$

E_4 in B simply indicates that $\underset{\sim}{x}$ could anything

E_3 in B & d $x_4 = 0$

Solving $Ax = b$ via Row Reduction:

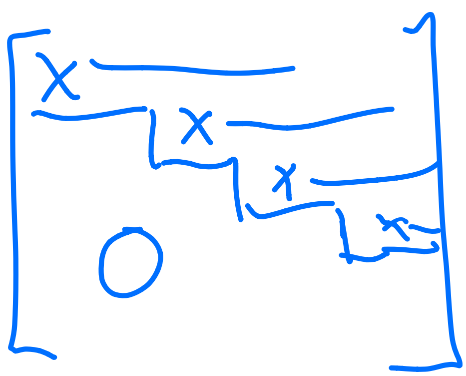
(Gaussian Elimination)

Algorithm

1. Begin by left-most column (this is pivot column)

2. Select a non-zero entry in that column. Use row ops to produce a equivalent matrix with zero in that entry.

3. Repeat in sub matrix, until row echelon form



4. Back substitution, working your way "up"

ex) $x_1 + 6x_2 + 3x_4 = 0$

$x_3 - 4x_4 = 5$

$x_5 = 7$

$$\left(\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

$x_5 = 7$ $\qquad$ $x_1 = -6x_2 - 3x_4$

$x_4 =$ free

$x_2 =$ free

//

Thm: (Existence)

A linear system is consistent iff the right most column of the augmented matrix is not a pivot column. (i.e. iff the echelon form of the equivalent (aug) matrix has no row of the form

$(\#)$ $\left[0\ 0 \cdots\cdots 0 \mid b\right]$ $b \neq 0$.

Overdetermined systems: More equations than unknowns.

Underdetermined systems: less equations than unknowns.

ex) $3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15$

$2x_2 - 4x_3 + 4x_4 + 2x_5 = -6$

$1x_5 = 4$

No row of the form $(\#)$ $\therefore$ 1 or more solutions possible.

$x_5 = 4$

$x_1 = 7 - 4x_3 + 3x_4$

$x_2 = -7 + 2x_3 - 2x_4$

underdetermined system

x_3, x_4 are free (parameters)

Vector Equations

Vector: is a quantity with magnitude and direction.

A vector may be real or complex valued:
We sometimes use the notation $\underset{\sim}{v} \in \mathbb{F}^n$ to mean that the vector $\underset{\sim}{v}$ can be either real or complex.

Vectors in $\mathbb{R}^n$

ex) $\underset{\sim}{x} \in \mathbb{R}^3$ means

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ or } \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

$\underset{\sim}{x} \in \mathbb{R}^n$

$$\underset{\sim}{x} = a_1\hat{e}_1 + a_2\hat{e}_2 \cdots a_n\hat{e}_n = \sum_{i=1}^{n} a_i\hat{e}_i$$

$$|\hat{e}_i| = 1$$

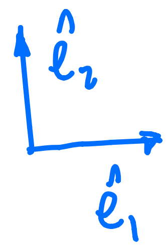
$\hat{e}_i$ are the "standard basis" in $\mathbb{R}^n$

$$\left.\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \hat{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right\} n \text{ long}$$

$\hat{e}_i \in \mathbb{R}^n$

ex) $\hat{e}_i \in \mathbb{R}^2$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mathbb{R}^2$



ex) $\hat{e}_i \in \mathbb{R}^3$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$