An Application of the Matrix Exponential

Consider the 1-parameter matrix

$$e^{At}$$

A is $n \times n$ matrix

t is a scalar (parameter)

Remind ourselves that, for α scalar

$$e^{\alpha t} = \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!}$$

$$= 1 + \alpha t + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \cdots$$

converges for all αt.

It is also true (Cayley-Hamilton, e.g.)

$$e^{At} = \mathbb{I} + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots,$$

$$= \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

$$= \frac{d}{dt}\left(\mathbb{I} + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots \right)$$

$$= 0 + A + A^2t + \frac{1}{2!}A^3t^2 + \cdots$$

$$= A\left[\mathbb{I} + At + \frac{1}{2!}A^2t^2 + \cdots \right]$$

$$= Ae^{At}$$

$$\frac{d}{dt} e^{At} = Ae^{At} \quad (\#)$$

Application: Solution of linear first order ordinary differential equations

IVP $\begin{cases} \text{ODE} & \dfrac{dy}{dt} = A y \quad t > 0 \\ \text{I.C.} & y(0) = \underset{\sim}{Y_0} \end{cases}$

$A \in \mathbb{R}^{n \times n}$, numbers

$y \in \mathbb{R}^n \quad \underset{\sim}{Y_0} \in \mathbb{R}^n$

if $A = \alpha$ a number then

$\begin{cases} \dfrac{dy}{dt} = \alpha y \\ y(0) = y_0 \end{cases}$ has a solution $y(t) = y_0 e^{\alpha t}$

Want $\underset{\sim}{y}(t)$ for $t \geq 0$ the solution of IVP

Suppose $y = e^{At} \underset{\sim}{W} \quad W \in \mathbb{R}^n$

Substitute into ODE

$$\frac{d(e^{At}\underset{\sim}{w})}{dt} = A e^{At}\underset{\sim}{w}$$

$$\frac{d e^{At}}{dt}\underset{\sim}{w} = A e^{At}\underset{\sim}{w}$$

using (#)

$$A e^{At}\underset{\sim}{w} = A e^{At}\underset{\sim}{w} \quad \checkmark$$

$$\therefore \left[\frac{de^{At}}{dt} - A e^{At}\right]\underset{\sim}{w} = 0 \text{ for } t>0$$

for $t=0$ require $\underset{\sim}{y}(0) = \underset{\sim}{Y}_0$ I.C

Guess that $\underset{\sim}{w} = \underset{\sim}{Y}_0$

$$\therefore \quad \underset{\sim}{y}(t) = e^{At}\underset{\sim}{Y}_0 \quad \text{for } t \geq 0$$

Compute $\{\lambda, \underset{\sim}{v}\}$ for A
e'values, e'vectors.

$$p(\lambda) = \det(A - \lambda I) = 0$$

$$(A - \lambda I)\underset{\sim}{v} = 0$$

$$y = c_1 \underset{\sim}{v}_1 e^{\lambda_1 t} + c_2 \underset{\sim}{v}_2 e^{\lambda_2 t} \cdots + c_n \underset{\sim}{v}_n e^{\lambda_n t}$$

so ODE

$$\frac{dy}{dt} = \lambda_1 c_1 v_1 e^{\lambda_1 t} + \lambda_2 c_2 \underset{\sim}{v}_2 e^{\lambda_2 t} + \cdots + \lambda_n c_n \underset{\sim}{v}_n e^{\lambda_n t}$$

since v_i are LI

so in each subspace of n-dimensional space

$$\lambda_i c_i v_i e^{\lambda_i t} = A c_i v_i e^{\lambda_i t}$$

$1 \leq i \leq n$

$$(A - \lambda_i I) c_i \underset{\sim}{v}_i e^{\lambda_i t} = 0$$

is automatically satisfied!

$\therefore$ (A) $\underset{\sim}{y} = e^{At}\underset{\sim}{y_0}$ is the solution

as is

(B) $\underset{\sim}{y} = \sum_{i=1}^{n} c_i \underset{\sim}{v_i} e^{\lambda_i t}$

$\underset{\sim}{v_i}$ & λ_i are the e'vectors/values of A.

at $t=0$ (A) & (B)

$$y(0) = \underset{\sim}{y_0} = \sum_{i=1}^{n} c_i \underset{\sim}{v_i}$$

$\Rightarrow$ tells us how to find c_i //

ex) Find solution $\begin{cases} \dfrac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x} & t>0 \\ \underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{cases}$

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

① Find $\{\lambda, v\}$ of A

$p(\lambda) = \det(A - \lambda I) = 0 = (1-\lambda)(\lambda-3)(\lambda+2) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = -2$

$\underset{\sim}{v_1} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \quad \underset{\sim}{v_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \underset{\sim}{v_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

② $\underset{\sim}{x} = c_1 \underset{\sim}{v_1} e^{\lambda_1 t} + c_2 \underset{\sim}{v_2} e^{\lambda_2 t} + c_3 \underset{\sim}{v_3} e^{\lambda_3 t}$

$\underset{\sim}{x}(t) = c_1 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-2t}$

$\underset{\sim}{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Let $\Phi = \begin{bmatrix} -1 & 1 & -1 \\ 4 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \Phi \underset{\sim}{c} \qquad \underset{\sim}{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$c = \Phi^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix} \frac{1}{3}$

$t \geq 0$

$$\underset{\sim}{x}(t) = \frac{5}{3}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{3t} - \frac{2}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t}$$

$$\underset{\sim}{x}(t) = \begin{bmatrix} -\frac{5}{3}e^{t} + 2e^{3t} + \frac{2}{3}e^{-2t} \\ \frac{4}{3}e^{t} - e^{3t} - \frac{2}{3}e^{-2t} \\ \frac{5}{3}e^{t} + 2e^{3t} - \frac{2}{3}e^{-2t} \end{bmatrix}$$