M is the matrix for the transformation T relative to the basis of $\mathbb{B}$ & $\mathbb{C}$.

$$M = \left[[T(b_1)]_{\mathbb{C}} \quad [T(b_2)]_{\mathbb{C}} \quad \cdots \quad [T(b_n)]_{\mathbb{C}} \right]$$ //

ex) $\mathbb{B} = \{b_1, b_2\}$ is the basis V

$\mathbb{C} = \{c_1, c_2, c_3\}$ basis for W

Suppose $T(b_1) = 3c_1 - 2c_2 + 5c_3$

$T(b_2) = 4c_1 + 7c_2 - c_3$

Find $M = \left[[Tb_1]_{\mathbb{C}} \quad [Tb_2]_{\mathbb{C}} \right]$

$$[T(b_1)]_{\mathbb{C}} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

$$[Tb_2]_{\mathbb{C}} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

if $\underset{\sim}{x} = \sum_{i=1}^{n} r_i \underset{\sim}{b}_i$ — $n=2$ here

$[x]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ here $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$T\underset{\sim}{x} = \sum_{i=1}^{n} r_i T b_i$ $n=2$ here

and $m=3$ here $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$$M[x]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 3r_1 + 4r_2 \\ -2r_1 + 7r_2 \\ 5r_1 - r_2 \end{bmatrix} = [T(x)]_{\mathcal{C}} \quad //$$

Linear Transformations from $V \to V$: only have $\mathcal{B}$

then M is the matrix for T relative to $\mathcal{B}$

$\Rightarrow$ "$\mathcal{B}$ matrix for T"

$$[T(x)]_{\mathcal{B}} = M[x]_{\mathcal{B}} \quad //$$

ex) $T = \frac{d}{dt}$ operator, consider $\mathbb{P}_2(t)$

(all poly's of degree at most 2)

$\mathcal{B}(\mathbb{P}_2(t)) = \{1, t, t^2\}$

$p \in \mathbb{P}_2 = a_0 1 + a_1 t + a_2 t^2$

$$[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$Tp = a_1 + 2a_2 t$$

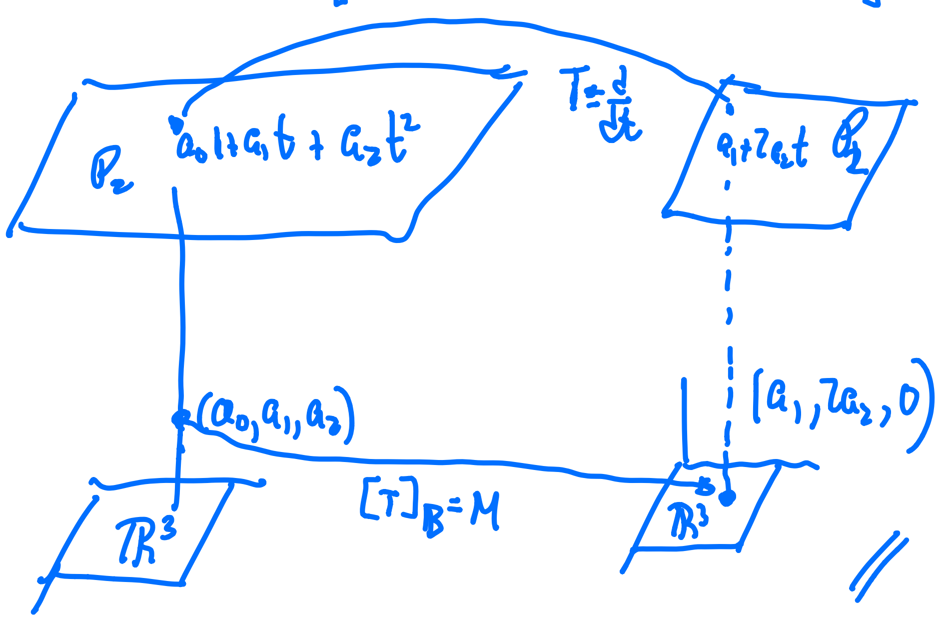
$$\begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} = M \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

by inspection $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$T(1) = 0 \qquad T(t) = 1 \qquad T(t^2) = 2t$

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = M = \begin{bmatrix} [T(1)]_{\mathcal{B}} & [T(t)]_{\mathcal{B}} & [T(t^2)]_{\mathcal{B}} \end{bmatrix}$$



E'vectors & Coordinate Transformations

Suppose T is diagonalizable.

$T(x) = Ax$, A is a matrix, an $n \times n$ matrix

If $A = V \Lambda V^{-1}$, the basis for $\mathbb{R}^n$ or $\mathbb{C}^n$ are the columns of V and the entries in Λ are the coordinates. The matrix Λ is also the M matrix!

So $V = [b_1\ b_2 \cdots b_n]$ columns with e'vectors

$$x = V[x]_{\mathbb{B}} \quad \text{and} \quad [x]_{\mathbb{B}} = V^{-1}x$$

i.e. if $A = V \Lambda V^{-1}$

$$[A]_{\mathbb{B}} = V^{-1}AV = \Lambda$$

$$M = [A]_{\mathbb{B}} \quad \& \quad [Ax]_{\mathbb{B}} = M[x]_{\mathbb{B}}$$

$$[A]_{\mathbb{B}} = \Big[[Tb_1]_{\mathbb{B}} \ \ [Tb_2]_{\mathbb{B}} \ \cdots \ [Tb_n]_{\mathbb{B}} \Big]$$

$$[A]_{\mathbb{B}} = \big[\, [Ab_1]_{\mathbb{B}} \;\; [Ab_2]_{\mathbb{B}} \;\cdots\; [Ab_n]_{\mathbb{B}} \,\big]$$

$$[A]_{\mathbb{B}} = \big[\, V^{-1}Ab_1 \;\; V^{-1}Ab_2 \;\cdots\; V^{-1}Ab_n \,\big]$$

$$= V^{-1}A\,[b_1\, b_2 \cdots b_n] = V^{-1}AV = \Lambda$$

//

ex) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ via a matrix A

Find a basis $\mathbb{B}$ for $\mathbb{R}^2$ with property that $[A]_{\mathbb{B}}$ is diagonal

for specificity $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

diagonalize A:

$p(\lambda) = \det(A - \lambda I) = 0 = (\lambda - 1)(\lambda - 3) = 0$

for $\lambda = 1$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 3$ $\quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$V = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad$ so $V^{-1}AV = \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$= [A]_{\mathbb{B}}$