Markov Chain

$$(\dagger) \quad \underset{\sim}{x}_{k+1} = A \underset{\sim}{x}_k$$

is a dynamic problem, in the general case.

If $\underset{\sim}{x}$ at "time" k is the vector of probabilities and A is a non-negative matrix with column sums $=1$

then $(\dagger)$ is a "Markov Chain" and A is known as a transition matrix.

A "Markovian" process is one in which the probability $\underset{\sim}{x}_k$ is fully determined by $\underset{\sim}{x}_{k-1}$ and $(\dagger)$.

The Stationary Probability vector

(or the steady state, if $(\dagger)$ is a dynamics problem, rather than a probability problem)

For $\underset{\sim}{x}_{k+1} = P \underset{\sim}{x}_k \quad (\text{✡})$

A steady state $\underset{\sim}{q}$ for (✡) is such that

$$(\exists) \quad P\underset{\sim}{q} = \underset{\sim}{q}$$

(a fixed point of P)

This fixed point $\underset{\sim}{q}$ will always exist if (✡) is $(\mp)$ a Markov Process.

Rewrite $(\exists)$ as

$$(P - \mathbb{I})\underset{\sim}{q} = \emptyset$$ so $\underset{\sim}{q}$ is the nontrivial

solution of the nullspace problem for $P - \mathbb{I}$.

This is a prescription for finding the stationary solution $\underset{\sim}{q}$.

If $(\cancel{*})$ represents a dynamic problem

$\underset{\sim}{x}_{k+1} = \underset{\sim}{x}_k$ means that $\underset{\sim}{x}_k$ does not change in "time", i.e. it is stationary //

Find stationary solution for

ex) $P = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ $\underset{\sim}{x}_0$ is known and has entries between 0 and 1 $\quad x_k \in \mathbb{R}^2$

$(*)$ $\underset{\sim}{x}_{k+1} = P\underset{\sim}{x}_k$.

$(*)$ is a Markov Chain.

then form $P - \mathbb{I} = \begin{bmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{bmatrix}$ let $\underset{\sim}{q} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$(P - \mathbb{I})\underset{\sim}{q} = \emptyset \rightarrow \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = \frac{3}{4}x_2$

augmented matrix $\quad x_2$ is free

$\therefore \underset{\sim}{q} = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} x_2$ pick $x_2 = 1$

$\underset{\sim}{q} = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

or $\quad \underset{\sim}{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, but $\underset{\sim}{q}$ represents probabilities, so the vector

$$3+4=7 \quad \therefore \quad \underset{\sim}{q} = \frac{1}{7}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

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If (#) is a Markov Chain, P is a transition matrix, then

$$x_{k+1} = Px_k, \quad k = 0, 1, \ldots$$

converges to $\underset{\sim}{x_k} = \underset{\sim}{q}$ as $k \to \infty$, regardless of the starting vector $\underset{\sim}{x_0}$.

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The Power Method (for estimating the largest eigenvalue of a matrix).

(the procedure can be converted into finding the smallest eigenvalue)

Spse A $n \times n$ matrix, want its largest eigenvalue.

Assume that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_n|$

let $\underset{\sim}{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2 \cdots + c_n \underline{v}_n$

$$(*) \quad A^k \underset{\sim}{x} = c_1 (\lambda_1)^k \underset{\sim}{v}_1 + \cdots\cdots c_n (\lambda_n)^k v_n$$
$$k = 1, 2 \ldots$$

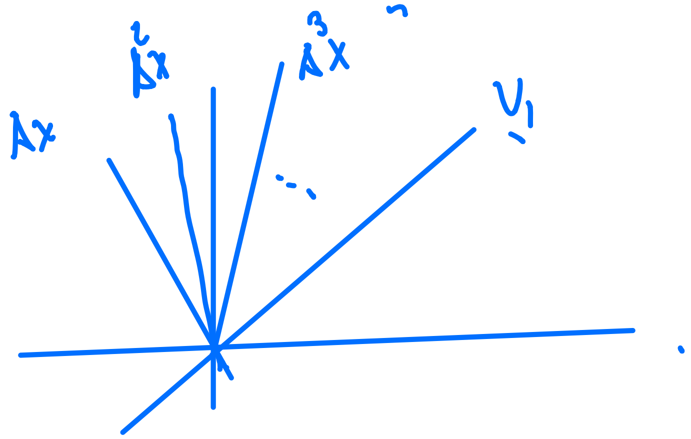
Assume $c_1 \neq 0$, divide B.S. of $(*)$:

$$\frac{1}{\lambda_1^k} A^k x = c_1 \underset{\sim}{v}_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k c_2 \underline{v}_2 + \left(\frac{\lambda_3}{\lambda_1}\right) c_3 \underset{\sim}{v}_3 \cdots + \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^k c_n v_n$$

Take $k \to \infty$

$$\frac{1}{\lambda_1^k} A^k \underset{\sim}{x} = c_1 \underline{v}_1$$

Which says that as $k \to \infty$ $A^k \underset{\sim}{x}$ points in the direction of $\underset{\sim}{v}_1$:



The Algorithm

1. Choose some $\underset{\sim}{x}_0$ so that its largest entry is 1.
set $\delta < 1$ to a small tolerance.

2. for $k = 0, 1, \ldots$

$$\underset{\sim}{y}_k = A \underset{\sim}{x}_k$$

$\mu_k = \max y_k$; μ_k is the entry in y_k whose absolute value is largest

$$x_{k+1} = \frac{1}{\mu_k} y_k$$

if $|x_{k+1} - y_k| < \delta$ exit.

output $\begin{cases} x_k & \text{the eigenvector} \\ \mu_k & \text{the eigenvalue} \end{cases}$ //

ex) Take $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$

$$\det(A - I\lambda) = 0 = (6-\lambda)(2-\lambda) - 5$$

$$\text{or } \lambda^2 - 8\lambda + 7 = 0$$

$$\lambda_1 = 7 \quad , \lambda_2 = 1$$

$$\underline{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad //$$

Now, use Power method: $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ initial guess.

$$A x_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \therefore \mu_0 = 5 \qquad x_1 = \frac{1}{5}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$A \underline{x}_1 = \begin{bmatrix} 8 \\ 1.8 \end{bmatrix} \therefore \mu_1 = 8$$

$$\underline{x}_2 = \frac{1}{8}\begin{bmatrix} 8 \\ 1.8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.225 \end{bmatrix}$$

$$A \underline{x}_2 = \begin{bmatrix} 7.125 \\ 1.430 \end{bmatrix} \quad \therefore \mu_2 = 7.125$$

$$\underline{x}_3 = \frac{1}{\mu_2}\begin{bmatrix} 7.125 \\ 1.450 \end{bmatrix}$$

$$A x_3 = \begin{bmatrix} 7.0175 \\ 1.4070 \end{bmatrix} \qquad \mu_3 = 7.0175$$

$\vdots$

$x_1 \to 7$ since μ_k $k \to \infty$ $\mu_k \to 7$.

you also get the evector associated λ_1:

$$\lim_{k\to\infty} \mu_k = 7$$

compare to

$$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

and $$\lim_{k\to\infty} A x_k = 7 \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$$

λ_1

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