Non-homogeneous Difference Equations

$P(E)\,y_k = z_k$, z_k is known.

General Solution technique

Variation of parameters.

ex) $y_{k+2} - 4y_{k+1} + 3y_k = -4k$ $\quad k=0,1,2,\dots$

$y_0 = 2$, $y_1 = 6$

$y_k = y_k^H + y_k^P$

$$\begin{cases} \text{A}\ P(E)\,y_k^H = 0 \text{ subject to } \underbrace{y_0=2,\ y_1=6}_{\text{I.C.}} \\ \text{B}\ P(E)\,y_k^P = -4k \end{cases}$$

$y_k^H \sim r^k$, substitute into A

$r^k(r^2 - 4r + 3) = 0$

if $r^k \neq 0 \Rightarrow (r^2 - 4r + 3) = 0$

$$(r-1)(r-3) = 0$$

roots are $r=1$, $r=3$

$y_k^H = a1^k + b3^k$ a, b constants.

To find a, b, use I.C.

$y_0^H = a1^0 + b3^0 = 2 = y_0$

$y_1^H = a1^1 + b3^1 = 6 = y_1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$a = 2$

$b = 2 - a = 0$

$\therefore y_H = 2 \cdot 1^k$ //

To solve B

(✯) $y_k^p = Ak^2$ is a guess

substitute (✯) in (B), find A

$$y_{k+2}^p - 4y_{k+1}^p + 3y_k^p = -4k \quad k=0,1\ldots$$

$$A\left[(k+2)^2 - 4(k+1)^2 + 3k^2\right] = -4k$$

Expand this, find $A=1$

$\therefore$ General solution

$$y_k = 2\cdot 1^k + k^2 \quad k=0,1,2\ldots$$

Unless you work in time series analysis or are developing auto-regressive models, you seldom see difference equations of order 2 and above. You more likely will exchange order of the difference equation for higher dimensions for the space of solutions.

1 order, high dimension difference Equations

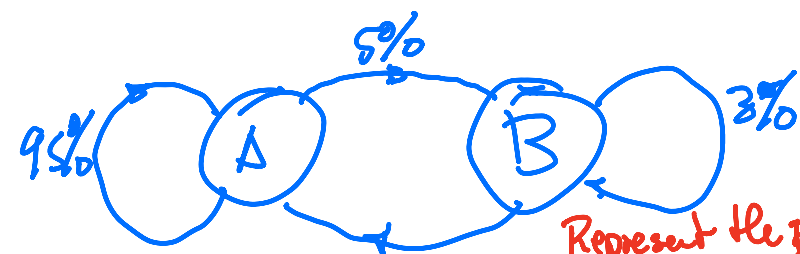
$$\underset{\sim}{x}_{k+1} = A\underset{\sim}{x}_k + \underset{\sim}{b}_k$$

known

$x_k \in \mathbb{R}^n \quad b_k \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$

starting vector $\underset{\sim}{x}_0$ is known.

ex)



Groups of people A, B

Represent the probabilities of 2 groups interacting

Model (Markov Model)

let k be the "time" variable.

$$\underset{\sim}{x}_{k+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \underset{\sim}{x}_k \qquad k = 0, 1 \ldots$$

Markov Transition Matrix

Column sum = 1

(entries are non negative)

The vector $\underset{\sim}{x}_k$ represents the probabilities of interactions for $A \to x_1$, $B \to x_2$

$$\underset{\sim}{x}_k = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_k$$

Suppose $\underset{\sim}{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$, let $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$

for example, $\underset{\sim}{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$. After m interactions:

$$\underset{\sim}{x}^m = A^m \underset{\sim}{x}_0 \text{. If } A = V \Lambda V^{-1}$$

$$\underset{\sim}{x}^m = V \Lambda^m V^{-1} \underset{\sim}{x}_0$$

Find Λ a 2x2 matrix with eigenvalues of A in the diagonal

1. Solve e'value problem

$$(A - \lambda I)\underset{\sim}{w} = 0$$

$$p(\lambda) = \det(A - \lambda I) = 0$$

roots of $p(\lambda)$ yield the e'values

$$p(\lambda) = \lambda^2 - 1.92\lambda + 0.92 = 0$$

$$\lambda_{1,2} = \frac{1.92}{2} \pm \frac{1}{2}\sqrt{0.0064} = \begin{cases} 1 \\ 0.92 \end{cases}$$

② Solve for associated eigenvectors:

For $\lambda_1 = 1$ $\quad (A - 1I)\,\underset{\sim}{w}_1 = \emptyset$

$$w_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

for $\lambda_2 = 0.92$ $\quad (A - 0.92I)\,\underset{\sim}{w}_2 = \emptyset$

$$\underset{\sim}{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = V \Lambda V^{-1}$$

$$V = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.92 \end{bmatrix}$$

$$x_m = V \begin{bmatrix} 1 & 0 \\ 0 & 0.92 \end{bmatrix}^m V^{-1} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$x_m = V \begin{bmatrix} 1^m & 0 \\ 0 & 0.92^m \end{bmatrix} V^{-1} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$\lim_{m \to \infty} x_m$ is a vector that lines up with e'vector associated with e'value = 1.

For Markov Transition Matrix problems as $n \to \infty$ the solution will line up (in direction with the e'vector associated with 1. Markov Problems always have 1 as an e'value, the e'values are all between [0,1].

Steady State Vector associated with a time dependent problem. Suppose

$$\underset{\sim}{x}_{k+1} = P \underset{\sim}{x}_k$$

P is a matrix, possibly dependent on k, the "time" variable

Also suppose $P \underset{\sim}{q} = \underset{\sim}{q}$ (not always possible)

(fixed point problem)

For $f(x) \in \mathbb{R}$, $x = f(x)$ is a fixed point problem