SIMILARITY TRANSFORMATIONS

Th'm: Let A, B be $n \times n$ matrices. They are similar if they have the same characteristic polynomial.

For A the characteristic polynomial is $\det(A - \lambda I) = p(\lambda) = 0$

If A, B are similar then the decomposition

$$V^{-1} A V = B$$

is possible, where V is an invertible matrix.

$V^{-1} A V$ is the similarity transformation.

Pf: $B - \lambda I = V^{-1} A V - \lambda I$

$$= V^{-1} A V - \lambda V^{-1} V$$

$$B - \lambda I = V^{-1}(A - \lambda I)V$$

$$\det(B - \lambda I) = \det\left[V^{-1}(A - \lambda I)V\right]$$

$$\det(B - \lambda I) = \det(V^{-1})\det(A - \lambda I)\det V$$

but $\det(V^{-1})\det(V) = 1$

$$\therefore \det(B - \lambda I) = \det(A - \lambda I) \quad //$$

$\det(A - \lambda I) = \det(B - \lambda I)$ does not necessarily imply that the e'values of A and B are the same //

DIAGONALIZATION of a square matrix is an application of similarity transformations

Factor $A = V \Lambda V^{-1}$ ($)

Λ is a diagonal matrix

$A = V \Lambda V^{-1}$ A nxn matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

λ_i are the eigenvalues of A.

V is composed of columns corresponding to e'vectors of A.

Not every nxn matrix can be diagonalized:
If n LI eigenvectors cannot be found then A does not have a diagonalization.

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$ for A $n \times n$ matrix

λ_i can be complex

If ($) is possible we say **A is diagonalizable**

The V is clearly made up of the e'vectors of A.

ex) $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ Find $A = V \Lambda V^{-1}$

or $AV = V\Lambda$

1. Find e'values of A

$$p(\lambda) = (\lambda - 1)(\lambda + 2)^2 = 0$$

$$\lambda_1 = 1, \ \lambda_2 = -2, \ \lambda_3 = -2$$

2. Find e'vectors of A

$\lambda_1 = 1$

$$\begin{aligned}
(1-1)x_1 + 3x_2 + 3x_3 &= 0 \\
-3x_1 - 4x_2 - 3x_3 &= 0 \\
3x_1 + 3x_2 + 0x_3 &= 0
\end{aligned}
\qquad v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$

$$\begin{aligned}
(1+2)x_1 + 3x_2 + 3x_3 &= 0 \\
-3x_1 - 3x_2 - 3x_3 &= 0 \\
3x_1 + 3x_2 + 0x_3 &= 0
\end{aligned}
\qquad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

v_1, v_2, v_3 are LI

For $V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

check that $AV = V\Lambda$

LINEAR DIFFERENCE EQUATIONS

Assume we have a sequence y_k $k=0,1,2\ldots$

Propose an operator E^n (displacement)

$$E^0 y_k = y_k$$

$$E^1 y_k = y_1, y_2 \ldots y_{k+1} \ldots$$

$$E^n E^m y_k = E^n[y_{k+m}] = y_{k+m+n}$$

etc

A linear difference equation (a recurrence equation)

$$P(E)\, y_k = z_k \qquad k=0,1,2\ldots$$

$P(E)$ is a polynomial of displacement operators, commonly with constant coefficient: $P(E) = a_n E^n + a_{n-1} E^{n-1} \dots + a_0 E^0$

$z_k = 0$ (homogeneous linear difference equation)

$z_k \neq 0$ (known) non-homogeneous problem,

ex) $P(E) = 6E^3 - 2E^2 + E^0$, $z_k = 0$ then

$P(E)\, y_k = 0$, a homogeneous equation:

$$(6E^3 - 2E^2 + E^0)\, y_k = 0$$

$$6y_{k+3} - 2y_{k+2} + y_k = 0 \qquad k = 0, 1, 2 \dots$$

$$y_{k+3} = \frac{1}{3} y_{k+2} - \frac{1}{6} y_k$$

need 3 starting values, say y_0, y_2, y_1

An application of diagonalization to solving

$$\underset{\sim}{x}_{k+1} = A\underset{\sim}{x}_k$$

A is a matrix of constants

$$\underset{\sim}{x}_k = A\underset{\sim}{x}_{k-1}$$

General Solution to $P(E)\,y_k = z_k$

is $y_k = y_k^H + y_k^P$

where $P(E)\,y_k^H = 0$

yields the homogeneous part

$P(E)\,y_k^P = z_k$

Solve the homogeneous problem:

$P(E)\,y_k = 0$

A good guess $y_k = r^k$ for some r.

ex) $\left(E^3 - 2E^2 - 5E^1 + 6E^0\right) y_k = 0$

(*) $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$

Assume $y_k = r^k$

(this guess works when roots aren't repeated) if it is a solution, it satisfies $(\circledast)$:

$$r^{k+3} - 2r^{k+2} - 5r^{k+1} + 6r^k = 0$$

$$r^k \left[r^3 - 2r^2 - 5r + 6 \right] = 0$$

$r=0$ is always a solution, but want nontrivial solution

$$r^3 - 2r^2 - 5r + 6 = 0$$ characteristic eq.

$$(r-1)(r+2)(r-3) = 0$$

3 roots: $1, -2, 3$

$$v_1 = 1^k \qquad v_2 = (-2)^k \qquad v_3 = 3^k$$

$$\therefore \; y_k = a\,1^k + b\,(-2)^k + c\,3^k \qquad k = 0, 1, \ldots$$

a, b, c are constants.

To find a, b, c, given y_k for 3 different k's.

In this example the roots are distinct & real.

Suppose roots are complex (conjugate pairs) $r_1 = \alpha + i\beta$ $\quad r_2 = \alpha - i\beta$

Suppose also r_3 is real

$$y_k = a r_3^k + b\alpha^k \cos(\beta k) + c\alpha^k \sin(\beta k)$$

is the assumed form of the solution...

For repeated roots, use "reduction of order"

Suppose r_1 is repeated

r_1^k is solution associated with this root.

$r_2 = u(k) r_1^k$

r_2 has to satisfy the difference equation.

This yields a difference equation for $u(k)$.

Reducing A Difference Equation to First Order (Normal Form)

Take $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$

Let $x_k = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$

$(\#)$ $$y_{k+3} = 2y_{k+2} + 5y_{k+1} - 6y_k$$

$$\underset{\sim}{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix} = \begin{bmatrix} 0 & y_{k+1} & 0 \\ 0 & 0 & y_{k+2} \\ \underbrace{-6y_k \quad 5y_{k+1} \quad 2y_{k+2}}_{\text{Using } (\#)} \end{bmatrix}$$

$(\exists)$ $$\underset{\sim}{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \underset{\sim}{x}_k = A\underset{\sim}{x}_k$$

Need to be able to solve (7). The general solution

$$\underset{\sim}{x}_k = C\, y_k$$

$$C = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & 0 \\ & & & 1 & \\ & 0 & & \ddots & 1 \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix}$$

i.e.

$$y_{k+n} + a_1 y_{k+n-1} + a_2 y_{k+n-2} \cdots a_{n-1} y_{k+1} + a_n y_k = 0$$

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One can solve (7) numerically. Sometimes one can find closed form solutions.

We're considering using DIAGONALIZATION

$$\underline{x}_{k-1} = A\underline{x}_{k-2}$$

$$\underline{x}_{k+1} = A(A\underline{x}_{k-1}) = A(A(A\underline{x}_{k-2}))\cdots$$

So generally $\boxed{\underline{x}_m = A^m \underline{x}_0}$

Suppose $A = V\Lambda V^{-1}$ is possible

$$\boxed{x_m = V\Lambda^m V^{-1}}$$

$$\Lambda^m = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

ex) $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ $V = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $V^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$$

Find $A^2 = (V\Lambda \underbrace{V^{-1})(V}_{I} \Lambda V^{-1})$

$$= V \Lambda^2 V^{-1}$$

$$A^3 = V\Lambda^3 V^{-1}$$

$$A^m = V \Lambda^m V^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^m & 0 \\ 0 & 4^m \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 5^m - 4^m & -5^m + 4^m \\ 2\cdot 5^m - 2\cdot 4^m & -5^m + 2\cdot 4^m \end{bmatrix}$$