* If 0 is an eigenvalue of a matrix A
$\Rightarrow$ A does not have an inverse.

* **Thm:** Let A be an $n \times n$ matrix.

If $v_1, v_2, \dots, v_r$ are e'vectors corresponding to DISTINCT eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of A, then $v_1, v_2, \dots, v_r$ are LI.

Pf: S'pse $\{v_1, v_2, \dots, v_r\}$ are LD. Without loss of generality, assume v_r can be written as a linear combination of $\{v_1, v_2, \dots, v_{r-1}\}$:

$$v_r = c_1 v_1 + c_2 v_2 \dots c_{r-1} v_{r-1}. \quad (1)$$

Multiply B.S. of (1) by A:

$$A v_r = A(c_1 v_1 + \dots c_{r-1} v_{r-1}) \text{ or}$$

$$\lambda_r v_r = c_1 \lambda_1 v_1 + \dots c_{r-1} \lambda_{r-1} v_{r-1} \quad (2)$$

Subtract $\lambda_r v_r$ from B.S of (2):

$$0 = c_1(\lambda_1 - \lambda_r) v_1 + c_2(\lambda_2 - \lambda_r) v_2 \dots c_{r-1}(\lambda_{r-1} - \lambda_r) v_{r-1}.$$

but by assumption $(v_1, \dots, v_{r-1})$ are LI

$\therefore$ $c_1(\lambda_1 - \lambda_r) = 0$, $c_2(\lambda_2 - \lambda_r) = 0$,

$\vdots$

$c_{r-1}(\lambda_{r-1} - \lambda_r) = 0$.

Since $\lambda_i - \lambda_j \neq 0$ for $i \neq j$ $\Rightarrow c_i = 0$ $i = 1, \dots, r-1$.

but by (1) $\Rightarrow v_r$ must be 0,

Which is not possible (a contradiction)

$\therefore$ $\{v_1, \dots, v_r\}$ cannot be LD. They must be LI.

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The Invertibility Theorem (continued)

Let A be $n \times n$ matrix. The following are equivalent:

* A cannot have 0 as an eigenvalue.
* $Col(A) = \mathbb{R}^n$

* $\mathrm{rank}(A) = n$
* $\mathrm{Null}(A) = \emptyset$
* $\dim(\mathrm{Null}(A)) = 0$
* Cols of A form a basis for $\mathbb{R}^n$ //

Finding Eigenvalues:

For the matrix A, an $n \times n$ matrix:

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

is the eigenvalue problem. Let

$$M \equiv A - \mathbb{I}\lambda, \quad \text{where} \quad \mathbb{I} = \mathrm{id}_n.$$

To find e'vectors $\underset{\sim}{x}$ associated with λ, $\lambda \in \mathbb{C}$: find the null space $\mathrm{Null}(M)$, i.e. find nontrivial $\underset{\sim}{x}$ such that $(A - \lambda\mathbb{I})\underset{\sim}{x} = 0$. //

To find λ, the eigenvalues:

let $p(\lambda) \equiv \det(A - \lambda I)$

a polynomial in λ.

To find e'values, set

$p(\lambda) = 0$, find roots. The roots are generally complex.

Why does this yield the e'values?

Want $M\underset{\sim}{x} = \phi \quad \underset{\sim}{x} \neq \phi$

so M has to have free parameters in its space of solutions.

i.e. M has L.D. vectors (the eigenvectors)

$\Rightarrow$ M is not invertible. So reasonable to suggest we find λ so that

$\det(M) = 0$. But need to be careful:

Appeal to Cramer's Rule: Suppose want to solve $M\underset{\sim}{x} = \underset{\sim}{u}$ $\quad \underset{\sim}{u} \neq \emptyset$.

$M \in \mathbb{R}^{n \times n}$ $\quad u \in \mathbb{R}^n$

$\underset{\sim}{x}$ is the solution $\in \mathbb{R}^n$. To find $\underset{\sim}{x}$, first write

$$\underset{\sim}{x} = \sum_{j=1}^{n} x_j \hat{e}_j$$ $\hat{e}_i$ is the standard basis.

$$M\underset{\sim}{x} = \sum_{j=1}^{n} m_j x_j \hat{e}_j = \underset{\sim}{u}$$

$$M = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}$$ m_i are columns of M.

$$(\star) \quad \sum_{j=1}^{n} m_j x_j = \underset{\sim}{u}$$

Let $M_k \equiv \begin{bmatrix} m_1 & m_2 & \cdots & m_{k-1}, & \underset{\sim}{u}, & m_{k+1} & \cdots & m_n \end{bmatrix}$

$$M_k = \left[m_1\ m_2 \cdots m_{k-1}, \sum_{j=1}^{n} m_j x_j , m_{k+1}, \ldots m_n \right]$$

using $(\cancel{*})$

Compute $\det(M_k)$

$$\det(M_k) = \sum_{j=1}^{n} x_j \det\left(m_1 \ldots m_{k-1}, m_j, m_{k+1} \ldots m_n\right) \quad (\dagger)$$

Note $\det(a_1, a_2 \ldots, a_n) = 0$ if $a_i = a_j$ $i \neq j$

if $A = [a_1\ a_2 \cdots a_n]$ a_i are columns of A.

So only 1 term in sum in $(\dagger)$ is non-zero:

$$\det(M_k) = x_k \det(M)$$

$$\therefore \; x_k = \frac{\det(M_k)}{\det(M)}$$

$\det(M) \neq 0$ leads to solution x_k, i.e. M has M^{-1}

x_k is the k^{th} entry of $\underset{\sim}{x}$.

THIS IS CRAMER'S RULE.

$\det(M_k) = \det(Adj(M))$

is computed using Laplace identity that related M_k to the cofactor expression.

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So, for $M = A - \lambda I$

then $\det(M) = 0$ is consistent with $M\underset{\sim}{x} = \emptyset$ having nontrivial solutions.

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