THE EIGENVALUE/EIGENVECTOR PROBLEM

Spse $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

if used as linear transformation, it will generally rotate and stretch an $\mathbb{R}^2$ vector

Try $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ & $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

as inputs:

$Au = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$ $\qquad$ $Av = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2v$

see that Au doesn't yield αu, α is a constant

However Av yields $2v$.

We'll find that there's a family of vectors $\{v_1, v_2 \ldots\}$ such that for some $A \in \mathbb{R}^{n \times n}$ and $v_i \in \mathbb{C}^n$

$$Av_i = \lambda_i v_i$$

λ_i is a number (possibly complex)

These v_i's and λ_i's are called eigenfunctions & eigenvalues of A

(linear operators defining vector spaces may have e'values & e'vectors, possibly complex)

An e'vector of an $n \times n$ matrix A is a nonzero vector $\underset{\sim}{v}$ such that $A\underline{v} = \lambda \underline{v}$, for some scalar (possibly complex) λ. If $\exists$ a non-trivial solution to $Av = \lambda v$ then we say v is an eigenvector of A.

For $M = A - \lambda \mathbb{I}$

$$A \in \mathbb{R}^{n \times n} \quad M \in \mathbb{C}^{n \times n}, \ \mathbb{I} \in id^{n \times n}$$

Form $M\underset{\sim}{v} = \emptyset \qquad (\ddagger)$

the non-trivial solutions to $(\ddagger)$ are the eigenvectors, associated with a particular λ.

(†) Gives a prescription for finding $\underset{\sim}{v}$'s, for each λ,

How to find λ's? later.

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ex) Is $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$?

Need to show that $A\underset{\sim}{u} = \lambda \underset{\sim}{u}$

λ is #

$$A\underset{\sim}{u} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -2\underset{\sim}{u}$$

$\therefore$ $\underset{\sim}{u}$ is an e'vector of A and -2 is its e'value.

is $\underset{\sim}{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$? $\quad A\underset{\sim}{u} = \begin{bmatrix} 4 \\ 18 \end{bmatrix} \neq \alpha \underset{\sim}{u}$ $\quad$ α a number.

$\therefore$ $\underset{\sim}{u}$ is not e'vector of A.

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ex) Show that $\lambda = 7$ is e'value of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

let $M = A - 7\mathbb{I} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$

Form $M\underset{\sim}{x} = \emptyset$ the argument matrix is

$$\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

then the general solution (nontrivial)

$$x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 \text{ free.}$$

$\therefore$ any vector proportional to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A //

The set of all nontrivial solutions of $(A - \lambda\mathbb{I})\underset{\sim}{x} = 0$ span $\text{Null}(A - \lambda\mathbb{I})$.

for $A \in \mathbb{R}^{n \times n}$ the vectors form a subspace of $\mathbb{R}^n$ called the eigenspace corresponding to a λ.

ex) $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has $\lambda = 2$ as e'value. Find the basis for the corresponding eigenspace:

Solve for nontrivial solutions of

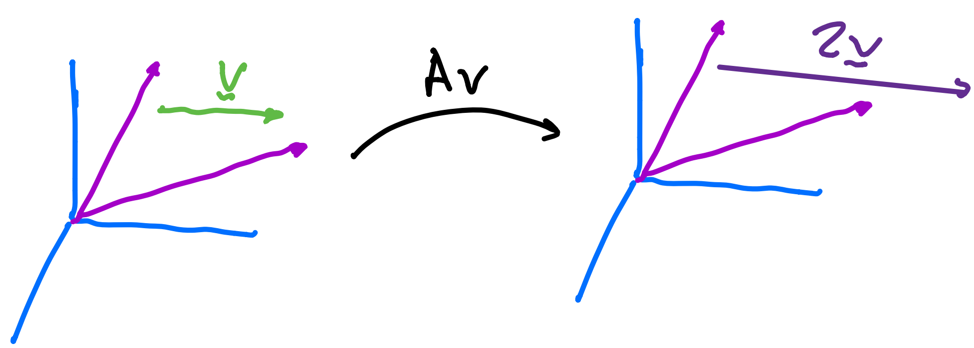
$(A - 2I)\underset{\sim}{x} = 0 \Rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \underset{\sim}{x} = 0$. The augmented matrix, after row ops, is

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$ x_2, x_3 are free

Basis for eigenspace associated with $\lambda = 2$ is

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^3$



How to find E'values? Special Case:

<u>Thm</u>: The e'values of a triangular matrix are the the zeros of $A-\lambda I$:

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & & & X \\ & a_{22}-\lambda & & \\ & 0 & \ddots & \\ & & & a_{nn}-\lambda \end{bmatrix}$$

So $(A-\lambda I)\underset{\sim}{x}=\emptyset$ has a solution $\underset{\sim}{x}\neq 0$

iff $(A-\lambda I)$ is zero in some sense: i.e.

iff $(A-\lambda I)\underset{\sim}{x}=\emptyset$ generates free variable(s).

Look at diagonal: It has at least 1 element where $a_{ii}-\lambda=0$. This will thus be the case //

ex) Find e'values of

$$A=\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

(upper triangular)

$\det(A-\lambda I) =$
$(3-\lambda_1)(0-\lambda_2)(2-\lambda_3)=0$

the $\det(A-\lambda I)=0$ is satisfied if

$$\lambda_1=3,\ \lambda_2=0,\ \lambda_3=2$$

these are the eigenvalues of A

(happen to be real and distinct)

ex) for $\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$ $\lambda_1=4$ $\lambda_2=1$ $\lambda_3=4$

they are real but 2 are unique.

$\lambda=4$ has multiplicity 2 e'value.

$\lambda=1$ is a simple e'value

ex) $A=\begin{bmatrix} 1+i\varepsilon & 1 \\ 0 & 1-i\varepsilon \end{bmatrix}$, $i=\sqrt{-1}$.

has 2 eigenvalues $1\pm i\varepsilon$, complex conjugates.

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