Matrix Notation

$$(\ast)\quad \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\boxed{\begin{array}{l}\text{Can also be written as} \\ \displaystyle\sum_{j=1}^{3} a_{ij} x_j = b_i \qquad i=1,2,3\end{array}}$$

the unknowns are x_1, x_2, x_3. Let $\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \underset{\sim}{x}$.

The coeffs in front of the $\underset{\sim}{x}$ components are

$$a_{ij} = \{a_{ij}\}_{i=1}^{3}\,{}_{j=1}^{3}$$

row column

the RHS $\begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix} = \underset{\sim}{b}$, known

So $(\ast)$ is written as $\boxed{\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} \quad (\star)}$

where $\underset{\sim}{A}$ is coefficient matrix.

Indicial Notation $(\star)$ can be

written as

$$\sum_{j=1}^{3} a_{ij} x_j = b_i \qquad i=1,2,3$$

or using "Einstein notation", where repeated indices are assumed summed:

$$a_{ij} x_j = b_i \qquad i=1,2,3$$

The Augmented Matrix

The system

$$A\underline{x} = \underline{b} \qquad (\bigstar)$$

is written in terms of the augmented matrix

$$(A|b)$$

. For example ($\not\bigstar$)

is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

For $A\underline{x} = \underline{b}$

Where $\underset{\sim}{x}$ is the unknown

We ask: can $\underset{\sim}{x}$ be found?

if found, is it unique?

if not found $\Rightarrow A\underline{x} = \underline{b}$ is "inconsistent".

Row Reduction (Gaussian Elimination)

ex) E_1: $x_1 - 2x_2 + x_3 = 0$

E_2: $2x_2 - 8x_3 = 8$

E_3: $-4x_1 + 5x_2 + 9x_3 = -9$

E_i is the name of each row

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$ is the augmented matrix.

We Generate an "Equivalent" system:

$A\underset{\sim}{x} = \underset{\sim}{b} \sim C\underset{\sim}{x} = \underline{d}$ are equivalent iff their solution $\underset{\sim}{x}$ is the same.

Row reduction generates an equivalent system that is easier to solve via backsubstitution: let's demonstrate:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$4E_1 + E_3 \to E_3$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

$\frac{1}{2}E_2 \to E_2$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

$3E_2 + E_3 \to E_3$

Equivalent system

$$\left(\begin{array}{ccc|c} \underline{1} & -2 & 1 & 0 \\ 0 & \underline{1} & -4 & 4 \\ 0 & 0 & \underline{1} & 3 \end{array}\right) \Longleftarrow C\underset{\sim}{x} = \underline{d}$$

the pivots are underlined.

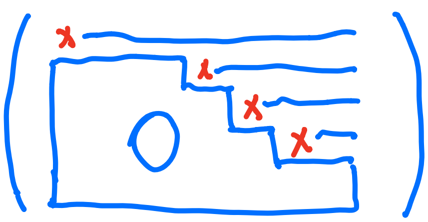
$$\left.\begin{aligned}&\text{from } E_3: && x_3 = 3\\ &\text{from } E_2: && x_2 = 4 + 4x_3 = 16\\ &\text{from } E_1: && x_1 = 2x_2 - x_3 = 29\end{aligned}\right\}\ \text{back substitution process.}$$

$$\underset{\sim}{x} = \begin{pmatrix} 29 \\ 16 \\ 3 \end{pmatrix} \quad \text{the solution.}$$

//

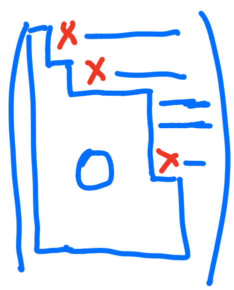
Row Echelon (Stair) Form:

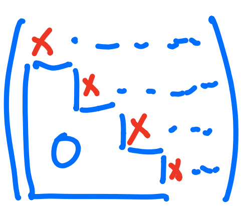
All of these matrices are in row-echelon form:



x are pivots

pivot columns have zeros below pivots.





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Suppose you find an equivalent matrix problem

$$A\underset{\sim}{x} = \underset{\sim}{b} \sim C\underset{\sim}{x} = \underset{\sim}{d}$$

$(C|d)$ has echelon form

Look at the back substitution issue:

Case (1) every row has a single pivot. Then

$$c_{nn}x_n = d_n \Rightarrow x_n = d_n/c_{nn}.$$

Case (2) You have row(s) that look like

$$0x_1 + 0x_2 + \dots + 0x_n = 0$$

$\Rightarrow$ x_i could all be anything and equation is consistent.

Case (3) A row might say

$$0x_j = d_i \neq 0 \Rightarrow \text{eq } \underline{\text{not}} \text{ consistent.}$$

Case (4) Row(s) might imply that

$x_j = a$, a number, and $x_j = b$, another number AND there are more equations than unknowns. (inconsistent equations).

ex) $E_1: 3x_1 + 6x_2 = -3$
$E_2: 5x_1 + 7x_2 = 10$ or $\left(\begin{array}{cc|c} 3 & 6 & -3 \\ 5 & 7 & 10 \end{array}\right)$

Produce an equivalent system:

$\frac{1}{3}E_1 \to E_1$ $\left(\begin{array}{cc|c} 1 & 2 & -1 \\ 5 & 7 & 10 \end{array}\right)$, $E_2 - 5E_1 \to E_2$ $\left(\begin{array}{cc|c} \underline{\underline{1}} & 2 & -1 \\ 0 & \underline{\underline{-3}} & 15 \end{array}\right)$

row echelon form

$=$ are pivots

Backsubstitution:

$-3x_2 = 15 \Rightarrow x_2 = -5$ $\quad x_1 = -1 - 2x_2 = 9$ //

The solution is consistent.

ex) Determine whether system is consistent:

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & 16 \end{array}\right]$$

$E_2 - 3E_1 \to E_2$

E_2: $(6-3h)x_2 = 16$

E_1: $x_1 + hx_2 = 4$

as long as $h \neq 2$ system is consistent

Row Reduction & Echelon Forms

An $A^{m \times n}$ is in echelon form if it has the following properties.

1. All non-zero rows are above any rows of zeros.
2. Each leading entry of a row is a column to the right of the leading entry of the row above it.

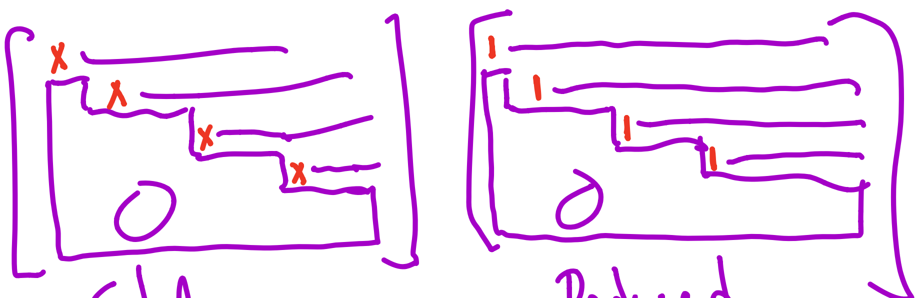
3. All entries in a column below a leading entry are zeros. ⇒ Echelon Form

If in echelon form & satisfies these additional properties, namely,

4. The leading entry in each non-zero row is 1

5. Each leading entry (1) is the only nonzero entry in its column.

⇒ Reduced Echelon Form



Echelon

Reduced Echelon