Determinants (Square Matrices)

Scalar function. For matrices of numbers it's a number.

Used in theoretical applications. Avoid computing a determinant of a large matrix using finite precision machines.

Determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

For 3x3

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

$$+ c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Generally if $A = \{a_{ij}\}$, $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} C_{ij}$

$(-1)^{i+j} C_{ij}$ is called the "cofactor."

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$C_{ij} = (-1)^{i+j} \det(c_{ij})$$

c_{ij} is obtained by covering the i^{th} row & j^{th} column (in this 3x3 case $\det c_{ij}$'s are 2x2).

ADJUGATE MATRIX: $\operatorname{adj}(A)_{ij} = (-1)^{i+j} C_{ji} \Rightarrow \operatorname{adj}(A) A = \det A \, I$

If matrix is triangular $\Rightarrow \det(A)$ = product of its diagonal entries.

ex) $\det \begin{bmatrix} a & & 0 & \\ & b & & \\ & & c & \\ & & & d \end{bmatrix} = \det A = a \cdot b \cdot c \cdot d$

$A \in \mathbb{R}^{4 \times 4}$

If any diagonal entry of the triangular

Square matrix is $0 \Rightarrow$ det of matrix $= 0$.

Th'm if $A \in \mathbb{R}^{n \times n}$

(i) If a multiple of row i is added to row j, producing a matrix B

$$\det(A) = \det(B)$$

(ii) If 2 rows are exchanged

$$\det(A) = -\det(B)$$

(iii) If the matrix B results from multiplying a row by a constant k,

$$\det(B) = k\det(A)$$

ex) $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$ Reduce A to row echelon equivalent

$$A \sim B = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{bmatrix} \qquad \det(A) = \det(B) = -15$$

$$C = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix} \qquad \det(C) = +15$$

(single row exchange)

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Thm: $\det(A^T) = \det(A)$

Thm: $\det(AB) = \det(A)\det(B)$

Thm: $\det(A) \neq 0 \Rightarrow \exists A^{-1}$

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Cramer's Rule

It applies to $\mathbb{R}^{n\times n}$

For 2x2 $\quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, known

Solve for $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ known

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

if $\det(A) \neq 0 \Rightarrow \exists A^{-1}$

$$x_1 = \frac{\det\begin{bmatrix} b_1 & \beta \\ b_2 & \delta \end{bmatrix}}{\det(A)}$$

$$x_2 = \frac{\det\begin{bmatrix} \alpha & b_1 \\ \gamma & b_2 \end{bmatrix}}{\det(A)}$$

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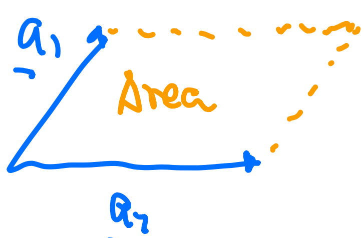
In $\mathbb{R}^2$

$$\det\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = |\det A| = \text{Area}$$



In $\mathbb{R}^3$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = A \in \mathbb{R}^{3\times 3}$$

$|\det A| =$ Volume of rhomboid

The determinants will be essential to the topic of spectral theory (theory of eigenvalues & eigenvectors).

USEFUL DETERMINANT PROPERTIES

* $\det(\mathbb{I}) = 1$
* $\det(A^T) = \det A$
* $\det(A^{-1}) = [\det(A)]^{-1} = \frac{1}{\det(A)}$
* $\det(AB) = \det(A)\det(B)$
* $\det(cA) = c^n \det(A)$; $A \in \mathbb{R}^{n \times n}$
* $\det(A+B) \geq \det(A) + \det(B)$
* if $\det(A) \neq 0 \iff A^{-1}$ exists

More examples of Vector Spaces and Subspaces

Quick Review

A subspace is a vector space inside another vector space.

A vector space represents a set of items we call "vectors". The set obeys the following properties

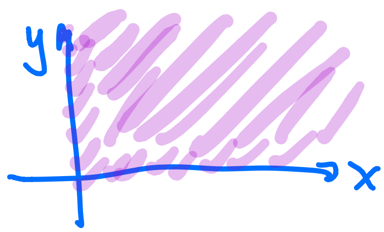
- Closed under addition
- Closed under multiplication by a scalar.

- The space has $\emptyset$
- $\exists$ a vector u associated with every v in the vector space s.t.

$$u+v=\emptyset$$

ex) Determine whether matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ can form a vector space.

ex) Let V be $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$



is not a vector space. For example, not closed under scalar multiplication:

i.e. $-1 \begin{bmatrix} x \\ y \end{bmatrix} \notin V$ //