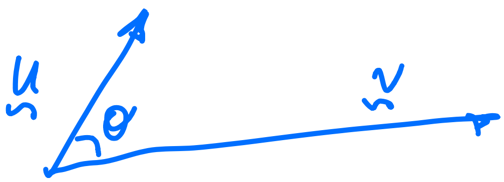
Orthogonality of 2 finite-dimensional vectors

Consider $\mathbb{R}^2$: Take $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^2$



The Dot Product is defined as

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$$

$\underset{\sim}{u}^T \underset{\sim}{v}$ for matrices
including vectors
(can be written as matrices)

If $\underline{u} \cdot \underset{\sim}{v} = 0$

then we say $\underset{\sim}{u} \perp \underset{\sim}{v}$
orthogonal

equivalently: $\underset{\sim}{u}^T \underset{\sim}{v} = 0$

Compute the vector spaces of matrix A

ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

Find $Col(A)$ via row reduction

$A \sim B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

take pivot column(s)

$col(A) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$r = rank(A) = dim(col(A)) = 1$

$Null(A)$ form $A\underset{\sim}{x} = 0$

& solve for $\underset{\sim}{x}$

$Null(A) : \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \underset{\sim}{x} = \emptyset \qquad \underset{\sim}{x} = (x_1, x_2)$

x_2 free, $x_1 = -2x_2$

$$\underset{\sim}{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 = \underset{\sim}{u}\ x_2$$

$$\mathrm{Null}(A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \underset{\sim}{u} \qquad \dim(\mathrm{Null}(A)) = \overset{2}{n} - \overset{1}{r} = 1$$

$$\mathrm{Row}(A) = \mathrm{Col}(A^T)$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$ row reduction of this

find $\mathrm{col}(A^T) = \mathrm{row}(A)$

$$\underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \dim(\mathrm{row}(A)) = r = 1$$

Find $\mathrm{Null}(A^T)$ To find this

$$A^T \underset{\sim}{y} = \emptyset$$

(interpret geometrically as finding all vectors y $\perp$ to columns of A.

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \underset{\sim}{y} = \emptyset \quad \text{find } y \qquad y = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\dim(\text{Null}(A^T)) = \overset{2}{m} - \overset{1}{r} = 1$$

ex) $$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$m = 4 \quad n = 6$

rref matlab (row echelon) to find

$$A \sim B = \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

gives pivot columns of A

(a) Find $\text{col}(A) = (a_1, a_3, a_6)$

$$= \left[\begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 15 \\ 18 \end{pmatrix} \right]$$

$$\dim(\text{col}(A)) = \text{rank} = 3$$

(b) Find Null Space of A

Set up $A\underset{\sim}{x}_h = 0 \quad (\ddagger)$

solve for $\underset{\sim}{x}$

Referring to $(\ddagger)$

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 = x_2 \\ x_3 = -2x_4 \\ x_4 = x_4 \\ x_5 = x_5 \\ x_6 = 0 \end{cases}$$

$$\underset{\sim}{x}_h = \underset{\sim}{N}_1 x_2 + \underset{\sim}{N}_2 x_4 + \underset{\sim}{N}_3 x_5$$

$$\underset{\sim}{x}_h = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_5$$

$$\dim(\text{Null}(A)) = n - r = 6 - 3 = 3$$

$$\text{Null}(A) = [\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3]$$

$Row(A) = col(A^T)$

$$A^T = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 3 & 6 & 0 & 6 \\ -2 & -5 & 5 & 0 \\ 0 & -2 & 10 & 8 \\ 2 & 4 & 0 & 4 \\ 0 & -3 & 15 & 18 \end{pmatrix}$$

could use $rref(A')$. Take the pivot rows of B:

$$row(A) = \left(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\dim(row(A)) = r = 3$$

Find $null(A^T)$

$$\dim(null(A^T)) = m - r = 4 - 3 = 1$$

$$rref(A^T) = \begin{pmatrix} \underline{1} & 0 & 10 & 0 \\ 0 & \underline{1} & -5 & 0 \\ 0 & 0 & 0 & \underline{1} \\ & \boxed{3 \times 4} & & \end{pmatrix}$$

Solve for y s.t. $A^T y = \emptyset$

$$y_1 = -10y_3$$
$$y_2 = 5y_3$$
$$y_3 = y_3$$
$$y_4 = 0$$

$$\underset{\sim}{y} = \begin{pmatrix} -10 \\ 5 \\ 1 \\ 0 \end{pmatrix} y_3$$

$$\text{null}(A^T) = \left(\begin{pmatrix} -10 \\ 5 \\ 1 \\ 0 \end{pmatrix} \right)$$