Column Space of a Matrix $A \in \mathbb{R}^{m\times n}$

Col(A) = is the column space of the $\mathbb{R}^{m\times n}$ matrix: all linear combinations of the columns of A.

The Col(A) = $\mathbb{R}^m$ iff Col(A) = span $\mathbb{R}^m$
Otherwise, it's a subspace of $\mathbb{R}^m$.

Null(A) = Null Space of $\mathbb{R}^{m\times n}$ matrix. All solutions $\underset{\sim}{x}$, to $A\underset{\sim}{x} = \emptyset$ (*)

Rmk: $\underset{\sim}{x} = \emptyset$ is always a solution to (*)

<u>Thm</u>: The Null(A) $\in \mathbb{R}^n$. Equivalently, the set of all solutions of (*), the m homogeneous linear equations in n unknowns (is a subspace $\mathbb{R}^n$) //

Basis of a Subspace: let H is a subspace of $\mathbb{R}^n$.

Then the basis for H is span(H) (with L.I vectors).

ex) Find a basis for the col(A)

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \end{array}$$

Once you found A in row echelon equivalent, the basis for col(A) = collection of pivot columns.

In this example A is in row echelon form.

The basis

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad a_5 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}.$$

Rmk: $a_3 = a_1 + 3a_2 \quad a_4 = 5a_1 + 2a_2$

$col(A) = span(a_1, a_2, a_5)$ //

Dimension & Rank of a Matrix (or of a subspace)

$\dim(H)$ = the dimension of subspace H corresponds to the number of vectors in the basis set of H.

By definition, $\dim(\phi) \equiv 0$.

$Rank(A)$ = rank of a matrix $A \equiv \dim(col(A))$

ex) $A = \{a_{ij}\}^{4\times5}$ row reduce to get

$$\begin{array}{c}\begin{bmatrix} \underline{x} & x & x & x & x \\ 0 & \underline{x} & x & x & x \\ 0 & 0 & \underline{x} & x & x \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \uparrow\ \uparrow\ \uparrow\ \end{array}$$

pivot columns $\therefore Rank(A) = r = 3$

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Thm If $A \in \mathbb{R}^{m\times n}$ then

$$\text{rank}(A) + \dim(\text{Null}(A)) = n$$

Recall that: to find Null(A) find all $\underset{\sim}{x}$ s.t.

$$A\underset{\sim}{x} = \phi$$

Thm: The Basis theorem: Let H be p-dimensional subspace of $\mathbb{R}^n$. Any LI set of exactly p elements belonging to H is a basis for H. Any set of p elements that span (H) is a basis of H. //

Thm: Invertible Matrix Theorem (Continued)

If $A \in \mathbb{R}^{n\times n}$ then the following statements are equivalent:

$\text{Col}(A) <\text{basis}(\mathbb{R}^n)$

$\text{Col}(A) = \mathbb{R}^n$

$\dim(\text{Col}(A)) = n$

$\text{rank}(A) = n$

$\text{Null}(A) = \emptyset$

$\dim(\text{Null}(A)) = 0$ //

There are 4 spaces associated with a Matrix (Linear) Transformation

In solving $A\underset{\sim}{x} = \underset{\sim}{b}$,

3 Problems Arise:

(i) Easiest: given A & $\underset{\sim}{x}$ find $\underset{\sim}{b}$

(ii) Intermediate: given A & $\underset{\sim}{b}$ find $\underset{\sim}{x}$.

(iii) Hard: given $\underset{\sim}{x}$ and $\underset{\sim}{b}$, find A.

For (ii) the general solution is

$$\underset{\sim}{x} = \underset{\sim}{x}_h + \underset{\sim}{x}_p$$

where $\underset{\sim}{x}_h$ can be not $\emptyset$.

$$A\underset{\sim}{x}_h = \emptyset \quad \& \quad A\underset{\sim}{x}_p = \underset{\sim}{b}$$ /

$$A : \mathbb{R}^n \to \mathbb{R}^m \qquad A \in \mathbb{R}^{m \times n}$$

For (i), $\underset{\sim}{A x} = \underset{\sim}{b}$ says $\underset{\sim}{a_1} x_1 + \underset{\sim}{a_2} x_2 \cdots + \underset{\sim}{a_n} x_n = \underset{\sim}{b}$

Where $A = \left.\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}\right\} m.$