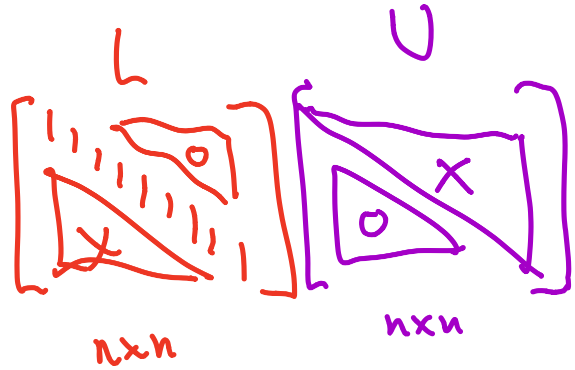
LU Decomposition

if $A \in \mathbb{R}^{n \times n}$

$A = LU =$ 

if we can reduce A into a matrix U by elementary row operations (each elementary row operation written in terms of an elementary matrix E_i) then, after p operations, say,

$$A = \underbrace{(E_p E_{p-1} E_{p-2} \cdots E_2 E_1)^{-1}}_{L} U$$

<u>Rmk:</u> Not every matrix will have an LU representation.

<u>Rmk:</u> The process of reducing A to U needs to

be done w/o row exchanges.

$$\therefore \boxed{E_p E_{p-1} \cdots E_2 E_1 A = U}$$

$$\therefore A = \underbrace{(E_p E_{p-1} \cdots E_1)^{-1}}_{L} U$$

$$\boxed{(E_p E_{p-1} \cdots E_2 E_1) L = I}$$

ex) Take $A \in \mathbb{R}^{4\times 5}$ and perform LU decomposition:

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \begin{array}{l} R_1 \to R_1 \\ 2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ 3R_1 + R_4 \to R_4 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} = L_1$$

pivot column

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ 3R_2 + R_3 \to R_3 \\ -4R_2 + R_4 \to R_4 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} = L_2$$

pivot column

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 \\ -2R_3 + R_4 \to R_4 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = L_3$$

pivot colum

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U \in \mathbb{R}^{m \times n}, \text{ so}$$

$$L = L_1 L_2 L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Rmk: Note that each L_i represents several row operation matrices //

SUBSPACES OF $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is any $H \in \mathbb{R}^n$ that has 4 fundamental properties:

(a) $\emptyset \in H$

(b) For each $\underline{u}, \underline{v} \in H$ $\quad \underline{u} + \underline{v} \in H$

(c) For each $u \in H$ $\quad \alpha u \in H$
α a scalar.

(d) $\exists$ a w s.t. $u + w = \emptyset \in H$
$w \in H$

" Vector subspace closed under scalar multiplication & vector addition" ex) if v_1 & $v_2 \in \mathbb{R}^n$

$H = \text{span}\{v_1, v_2\}$

H is a subspace of $\mathbb{R}^n$

Take $u = s_1v_1 + s_2v_2$ $v = t_1v_1 + t_2v_2$

$u+v = (s_1+t_1)v_1 + (s_2+t_2)v_2$

let c be a constant

$cu = cs_1v_1 + cs_2v_2$ a linear combination of v_1 & v_2 which is in span(v_1, v_2)

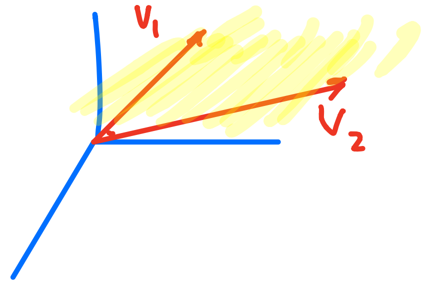
$\exists$ constants c_1, c_2 s.t.

$u = s_1v_1 + s_2v_2$

$v = c_1v_1 + c_2v_2$

$u+v = \emptyset$ //

ex) A familiar subspace of $\mathbb{R}^3$ is $\mathbb{R}^2$, for example, the plane spanned by $\underline{v}_1$ & $\underline{v}_2$:



ex) Let $V = \mathbb{R}^3$ let $ax_1 + bx_2 = c$ $c \neq 0$

$V = ax_1 + bx_2 - c$

Try to satisfy zero property:

$\emptyset = ax_1 + bx_2 - c$ cannot be satisfied.

$u = tv$ t, is constant

ok.

$\Rightarrow$ V is not a subspace of $\mathbb{R}^3$ //

ex) Take the set of all integers $\mathbb{Z}$. This is not a vector space. $\mathbb{Z}$ is not closed under scalar multiplication:

Take $5 \in \mathbb{Z}$. But $\pi 5 \notin \mathbb{Z}$ for example. //

Column Space & Null Space of a Matrix

The problem

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

Has 4 subspaces:

$$\begin{array}{ll} \text{Col}(A) & \text{Null}(A) \\ \underbrace{\text{Row}(A)}_{\text{Col}(A^T)} & \text{Null}(A^T) \end{array}$$