Consider the following variation of the Leontief Problem:

$$(\ast)\begin{cases} \underset{\sim}{x}_{n+1} = C\underset{\sim}{x}_n + \underset{\sim}{d} & n = 0,1,2,\dots \\ \underset{\sim}{x}_0 \text{ known} \end{cases}$$

n represents the year, say.

Question we may be asking:

"What will be the supply 5, 10, etc years from now, to meet the yearly demand $\underset{\sim}{d}$?"

Using $(\ast)$

$$x_n = Cx_{n-1} + d$$

$$x_{n-1} = Cx_{n-2} + d$$

$$\vdots$$

$$\therefore\ x_{n+1} = C\left(Cx_{n-1} + d\right) + d$$

$$= C\left(C\left(Cx_{n-2} + d\right) + d\right) + d$$

$$= C\left(C\left(C\left(x_{n-3} + d\right) + d\right) + d\right) + d$$

if we take $\underset{\sim}{x}_0 = \underset{\sim}{d}$ (for example)

$$\underset{\sim}{x}_n = \sum_{i=0}^{n} C^i \underset{\sim}{d} \qquad (\dagger)$$

The Geometric Series

Consider

$$x_n = \sum_{i=0}^{n} a^i \quad , \text{where } a \text{ is a number}$$

or $\quad x_n = 1 + a + a^2 + a^3 \cdots + a^n \qquad (*)$

Multiply $(*)$ by $1-a$

$$(1-a)(1+a+a^2 \cdots a^n) = 1 - a^{n+1}$$

Hence

$$1+a+c^2+\cdots a^n = \frac{1-a^{n+1}}{1-a}$$

or $\quad \displaystyle\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$, provided $|a| \neq 1$.

Further, suppose $|a|<1$, then

$$\sum_{i=0}^{\infty} a^i = \lim_{n\to\infty} \sum_{i=0}^{n} a^i = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}$$

Back to (‡):

$$\sum_{i=0}^{n} C^i = (I-C)^{-1}(I-C^{n+1}), \text{ provided } |C| \neq 1$$

If $|C|<1$ (if column sum <1)

$$\sum_{i=0}^{\infty} C^i = (I-C)^{-1}$$

So for this case

$$\underset{\sim}{x} = C\underset{\sim}{x} + \underset{\sim}{d}$$

$$(I-C)\underset{\sim}{x} = \underset{\sim}{d}$$

$$\underset{\sim}{x}^* = (I-C)^{-1}\underset{\sim}{d} = (I+C+C^2+C^3\dots)\underset{\sim}{d}$$

So we need to invert $(I-C)$

PARTITIONED MATRICES

It is often the case that a matrix has structure, typically block structure. One can make use of this structure.

The following is a typical example:

Consider the $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$

where each A_{ij} is a block

We want to multiply $A\,B$ (the multiplication has to make sense)

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

where B_1, B_2 are blocks