Show that if AB is invertible, so is A:

(cannot assume that A, B are each invertible).

If AB is invertible $\Rightarrow (AB)W = I$

$A(BW) = I \Rightarrow (BW) = A^{-1}$

Since A is square. //

Suppose $A \in \mathbb{R}^{5\times5}$ and $A\underline{x} = \underline{b}$ is consistent $\forall\, b \in \mathbb{R}^5$. Is it possible that for some $\underline{b}$, $A\underline{x} = \underline{b}$ has more than 1 solution?

No. $\underset{\sim}{b} \in \mathrm{col}(A)$ and $\mathrm{col}(A) = \mathrm{span}(\mathbb{R}^5)$

$\Rightarrow$ the solution is unique. //

The following statements are ALL TRUE OR ALL FALSE:

Thm: Let $A \in \mathbb{R}^{n \times n}$, the following are equivalent:

(i) A is invertible

(ii) A is row equivalent to I

(iii) A has n pivots

(iv) $A\underline{x} = \underline{0}$ has only trivial solution

(v) Col(A) are LI

(vi) $x \to Ax$ is 1 to 1 map.

(vii) $A\underline{x} = \underline{b}$ has at least one solution for each $\underline{b} \in \mathbb{R}^n$.

(viii) Col(A) = span($\mathbb{R}^n$)

(ix) $A\underline{x}$ maps $\mathbb{R}^n \to \mathbb{R}^n$

(x) $\exists$ $CA = I$ and $AD = I$, $D = C$.

(xi) A^T is invertible.

//

Applications, Continued

CHEMICAL REACTIONS

Consider the simplest of reactions: it conserves energy and mass. We want to write down the balance equation for

$$CH_4 + O_2 \to CO_2 + H_2O$$

We denote a vector

$$\begin{bmatrix} C \\ H \\ O \end{bmatrix}, \text{ so that}$$

$$CH_4 : \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \quad O_2 : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad CO_2 : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad H_2O : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The balance equation is

$$x_1 (CH_4) + x_2 (O_2) - x_3 (CO_2) - x_4 (H_2O) = 0$$

or

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\underset{\sim}{x} = [x_1, x_2, x_3, x_4]^T$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Hence $A\underset{\sim}{X} = 0$ can be solved using row reduction:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right]$$

$\therefore\ x_1 = \frac{1}{2}x_4,\ x_2 = x_4,\ x_3 = \frac{1}{2}x_4$

pick $x_1 = 1 \Rightarrow x_2 = 2,\ x_3 = 1,\ x_4 = 2$

$\therefore\ CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$ //

The Leontief I/O Model

This is the problem of supply and demand, for a closed economy (nothing gets imported/exported/destroyed/created)

Let $\underset{\sim}{x} \in \mathbb{R}^n$ "production" or "supply"

dimensional units are: [number of units]/[time]

Let $\underset{\sim}{d} \in \mathbb{R}^n$ "consumer demand"
or "target production"

$$(\dagger) \quad \underset{\sim}{x} = C\underset{\sim}{x} + \underset{\sim}{d}$$

$C \in \mathbb{R}^{n \times n}$

$C\underset{\sim}{x}$ represents "intermediate demand"
or "resource demand"

We construct C from data: for example
consider the following table

Purchased From:	Input units consumed/unit output		
	M	A	S
M (Manufacturing)	0.50	0.40	0.20
A (Agricultural)	0.20	0.30	0.10
S (services)	0.10	0.10	0.30

Let $\underset{\sim}{c}_i \in \mathbb{R}^{n\times 1}$ $i=1,2,\dots,n$

be the unit consumption vectors representing input needed/unit of output for each sector. The sectors are M, A, S above.

From table $c_1 = \begin{bmatrix} 0.50 \\ 0.20 \\ 0.10 \end{bmatrix}$ $c_2 = \begin{bmatrix} 0.40 \\ 0.30 \\ 0.10 \end{bmatrix}$ $c_3 = \begin{bmatrix} 0.20 \\ 0.10 \\ 0.30 \end{bmatrix}$

How would you use c_i? Could be asking

What number is consumed by A if 100 units are to be produced?

It'd be $100 c_2 = 100 \begin{bmatrix} 0.40 \\ 0.30 \\ 0.10 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 10 \end{bmatrix}$ //

The intermediate demand $C\underset{\sim}{x}$ is thus $\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \underset{\sim}{x}$

where c_1, c_2, c_3 are the unit consumption column vectors.

The typical question is "Given the

demand $\underset{\sim}{d}$ determine the supply $\underset{\sim}{x}$:

Using (†) $\quad \underset{\sim}{x} - C\underset{\sim}{x} = \underset{\sim}{d}$

or $(\mathbb{I} - C)\underset{\sim}{x} = \underset{\sim}{d}$

$\therefore \quad \underset{\sim}{x} = (\mathbb{I} - C)^{-1}\underset{\sim}{d}$

is the supply //

Suppose $\underset{\sim}{d} = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$ and $C = \begin{bmatrix} 0.50 & 0.40 & 0.20 \\ 0.20 & 0.30 & 0.10 \\ 0.10 & 0.10 & 0.30 \end{bmatrix}$

Find $\underset{\sim}{x}$.

let $M = \mathbb{I} - C \quad \therefore \quad \underset{\sim}{x} = M^{-1}\underset{\sim}{d}$

using row reduction:

$[M \,\vdots\, \underset{-}{d}]$ yield $\underset{\sim}{x}$

or find M^{-1} via row reduction

$[M \mid \mathbb{I}] \sim [\mathbb{I} \mid M^{-1}]$

$$M^{-1} = \begin{bmatrix} 2.9630 & 1.8519 & 1.1111 \\ 0.9259 & 2.0370 & 0.5556 \\ 0.5556 & 0.5556 & 1.6667 \end{bmatrix}$$

to 5 significant digit accuracy.

So $\underset{\sim}{x} = M^{-1}\underset{\sim}{d} = \begin{bmatrix} 226 \\ 119 \\ 78 \end{bmatrix}$

This is a unique (feasible, since non-negative) solution //