INVERSE OF A MATRIX

The goals of this section are to understand the mathematical & geometric meaning of the inverse of a matrix. We'll also compute some of these, however, computing inverses of matrices is very ardvous and numerically unstable, in the general case. You will learn how to take inverses of matrices in a numerical analysis class, not in this class.

Given a matrix $A \in \mathbb{R}^{n \times n}$ and another matrix $C \in \mathbb{R}^{n \times n}$ such that

$$CA = AC = I_n \quad (*)$$

We say that C and A are inverses of each other.

A matrix $A \in \mathbb{R}^{n \times n}$ for which there exist no C satisfying (*) is said to be noninvertible (or singular).

If (*) is satisfied we denote

$$\boxed{C = A^{-1}}$$

and we call it the inverse of A. //

Before proceeding, let's put some context on this issue:

Take a case you're familiar:

Let $a \in \mathbb{R}$ be a 1×1 matrix

(*) says that if $\exists c$ such that

$ac = ca = 1$ then c is the inverse of a. Well, if $a \neq 0$ $c = 1/a = a^{-1}$.

Looks straightforward... so if $a \neq 0$ then $c = 1/a$. What's the big deal?

Let's try $A \in \mathbb{R}^{2 \times 2}$, the next least complicated:

Clearly if $A = 0$ then (*) is not

attainable by any C. In the 1D dimension invertibility is a problem for the matrix $a=0$. In 2D we can write an arbitrarily large number of matrices that do not have an inverse and the entries in the matrix don't have to be zero.

Examples you're already familiar with are

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$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$ (the projection matrix),

$$A=\begin{bmatrix}1&0\\1&0\end{bmatrix},\quad A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$$

and scalar multiples of these.

In $\mathbb{R}^{n\times n}$ $n>1$ this situation is not exceptional as it is in $\mathbb{R}^{1\times 1}$.

The inverse of a 2x2 matrix is easy to find (and worth memorizing):

if $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$; $A^{-1}=\dfrac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

where $\det(A)=ad-bc$.

So A^{-1} fails to exist when $\det(A) = 0$.

This prompts the question, what properties or characteristics are common to all singular matrices? We'll get to this, but not now. //

For now, we can answer a more basic question: Why do we care whether A^{-1} exists?

Here are 2 practical reasons to care:

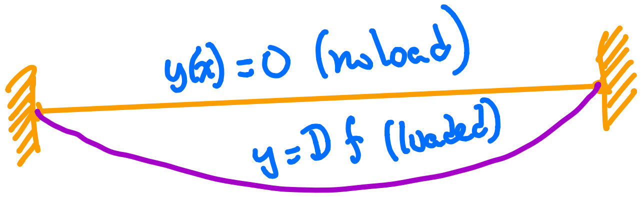
① if $A\underset{\sim}{x} = \underset{\sim}{b}$, where $\underset{\sim}{x}, \underset{\sim}{b} \in \mathbb{R}^n$, $\underset{\sim}{b}$ known, $A \in \mathbb{R}^{n \times n}$ known, we see that

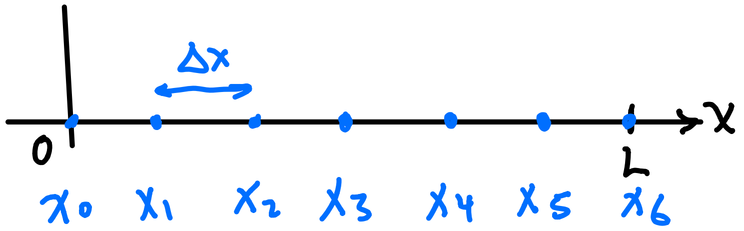
$$A^{-1}A\underset{\sim}{x} = A^{-1}b \Rightarrow \underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

gives us the unknown $\underset{\sim}{x}$.

② if $\underset{\sim}{b} = T\underset{\sim}{x}$ is a (linear) transformation, we might like to "undo" the transformation to find $\underset{\sim}{x} = T^{-1}\underset{\sim}{b}$. //

ex) Hooke's law relates the deflection of a beam $y(x)$ under the action of a force (load) $f(x)$





Discretized:

Let $y_i = y(x_i)$:

$y_0 = 0$, $y_6 = 0$ The beam is clamped at $x=0, x=L$.

So we know these deflections.

$D = D(x)$ is a material property and it is known.

It is discretized and approximated by a matrix $A \in \mathbb{R}^{5\times5}$, and it is given

The load at each x_i is f_i, $i=1,2,3,4,5$.

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ and $\underset{\sim}{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$ then,

$$\underset{\sim}{y} = A \underset{\sim}{f}$$

if we know $\underset{\sim}{f}$, we can find y.

if we know y, we can find the load $\underset{\sim}{f}$:

$$\underset{\sim}{f} = A^{-1} y \qquad //$$

PROPERTIES OF A^{-1}:

Provided A^{-1}, B^{-1} exist,

(1) $(A^{-1})^{-1} = A$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

(3) if A^{-1} exists so does the inverse of A^T

(3') $(A^T)^{-1} = (A^{-1})^T$

Reminder A^T is the transpose of A. It exists whether A is invertible or not.

If $A = (a_{ij})$ then $A^T = (a_{ji})$

HOW DO WE FIND A^{-1}? The painful and naive way is to use... ROW REDUCTION (as I said before, there are less painful ways to do it... take a numerical analysis course!)

def: An invertible matrix has a A^{-1}

The Row Reduction Algorithm

Given $A \in \mathbb{R}^{n \times n}$, invertible, form the augmented system

$$\left[A \,\middle|\, \mathbb{I}_n \right]$$

then, by row operations find

$$\left[\mathbb{I}_n \,\middle|\, C \right]$$

$C = A^{-1}$

//

Why does this work? The above can be viewed as solving n simultaneous equations

(‡) $A\underset{\sim}{x}_1 = \hat{e}_1$, $A\underset{\sim}{x}_2 = \hat{e}_2$, . . . $A\underset{\sim}{x}_n = \hat{e}_n$.

Why can be written as

$$\left[A \,\middle|\, \hat{e}_1\, \hat{e}_2 \cdots \hat{e}_n\right] = \left[A \,\middle|\, \mathbb{I}_n\right].$$

The solution of $\underset{\sim}{x}_i$ in (‡) are the columns of A^{-1}. Hence

$$\left[A^{-1}A \,\middle|\, A^{-1}\mathbb{I}\right] = \left[\mathbb{I}_n \,\middle|\, A^{-1}\right] \quad //$$

ex) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ Find A^{-1}

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \underset{E_1 \leftrightarrow E_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \cdots$$

$$\sim \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \,\middle|\, \underbrace{\begin{matrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{matrix}}_{A^{-1}} \right] \quad //$$

confirm (by hand) that you can recreate this result