Approximating Solutions to ODE's via Finite Differences (continued)

In our last lecture we derived

$$\begin{cases} \underset{\sim}{x}_{i+1} = \underset{\sim}{x}_i + hA\underset{\sim}{x}_i, & i = 0, 1, 2\ldots \\ \underset{\sim}{x}_0 = \underset{\sim}{X}_0, \text{ known.} \end{cases}$$

So $\underset{\sim}{x}_{i+1} = \mathbb{I}_n \underset{\sim}{x}_i + hA\underset{\sim}{x}_i = (\mathbb{I}_n + hA)\underset{\sim}{x}_i$

$(\dagger)$ $\quad \underset{\sim}{x}_{i+1} = M\underset{\sim}{x}_i \qquad M = \mathbb{I}_n + hA$ a constant matrix.

Note: $\underline{x}_i = M\underline{x}_{i-1}$

$x_{i-1} = Mx_{i-2}$

etc

So $(\dagger)$ $\underset{\sim}{x}_{i+1} = M(Mx_{i-1}) = M(M(Mx_{i-2}) \ldots$

By induction:

$$(\$)\quad \underset{\sim}{x}_n = M^n \underset{\sim}{x}_0 \qquad n=1,2\ldots$$

Note that ($) is familiar to us:

if we call $\underset{\sim}{x}_n = \underset{\sim}{z}$, $B = M^n$, $\underset{\sim}{x}_0 = \underset{\sim}{c}$

then ($) is of the form

$$\underset{\sim}{z} = B\underset{\sim}{c},$$

A linear transformation, with known

B and $\underset{\sim}{c}$.

MATRIX OPERATIONS

Thm: Let A, B, C be same-sized matrices $\mathbb{R}^{m\times n}$

(a) $A+B = B+A$

(b) $(A+B)+C = A+(B+C)$

(c) $A+\emptyset = A$

(d) $\alpha(A+B) = \alpha A + \alpha B$

(e) $(\alpha+\beta)A = \alpha A + \beta A$

(f) $\alpha(\beta A) = (\alpha\beta)A$

There are many ways to define the operation of multiplication in vector spaces. Here we focus on the simplest "matrix multiplication", as described below:

If $A \in \mathbb{R}^{m \times n}$ $B \in \mathbb{R}^{n \times p}$ $A = [a_{ij}]$, $B = [b_{ij}]$

$$AB = \sum_{j=1}^{n} a_{ij} b_{jk}$$

ex) $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

$A \in \mathbb{R}^{2\times2}$ $B \in \mathbb{R}^{2\times3}$

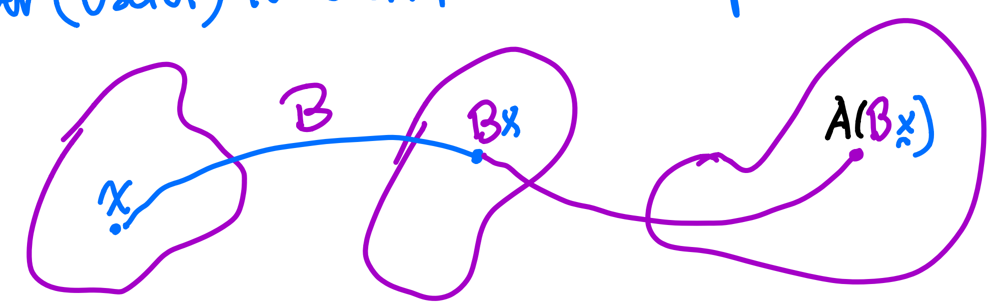
$\therefore AB \in \mathbb{R}^{2\times3}$

$$B \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}}_{AB}$$

Another (useful) Take on Matrix Multiplication:

$AB\underset{\sim}{x}$



So $(AB)x = Cx$

$$= \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix} \underset{\sim}{x}$$

columns of C

"Multiplication AB is a composition of 2 linear transformations"

Going back to the example

ex) $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

$A \in \mathbb{R}^{2\times 2}$ $\quad B \in \mathbb{R}^{2\times 3}$

$\therefore AB \in \mathbb{R}^{2\times 3}$

$$\begin{array}{ccc} & & B\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \\ A\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} & & \underbrace{\begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}}_{AB} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix} \end{array}$$

$$B = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$$

$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$ $\quad Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$

$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$

//