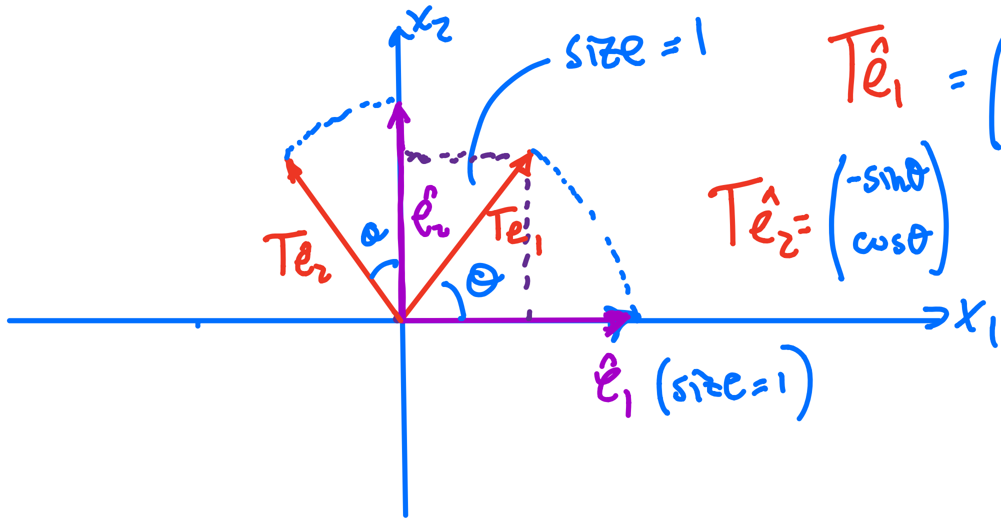
DERIVING THE ROTATION TRANSFORMATION IN $\mathbb{R}^2$



$T\hat{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$T\hat{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

Use standard basis for the derivation

Find $T\hat{e}_1$

Find $T\hat{e}_2$

let $\underset{\sim}{x} = x_1\hat{e}_1 + x_2\hat{e}_2$

then

$T\underset{\sim}{x} = x_1 T\hat{e}_1 + x_2 T\hat{e}_2$

$= \left[T\hat{e}_1 \; T\hat{e}_2 \right] \underset{\sim}{x}$

$\therefore \quad T\underset{\sim}{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underline{x}$

//

Existence & Uniqueness

Def: The mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if each $\underset{\sim}{b} \in \mathbb{R}^m$ is the image of at least one $\underset{\sim}{x} \in \mathbb{R}^n$.

Def: The mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is one to one if for each $\underset{\sim}{b} \in \mathbb{R}^m$ is the image of at most one $\underset{\sim}{x} \in \mathbb{R}^n$.

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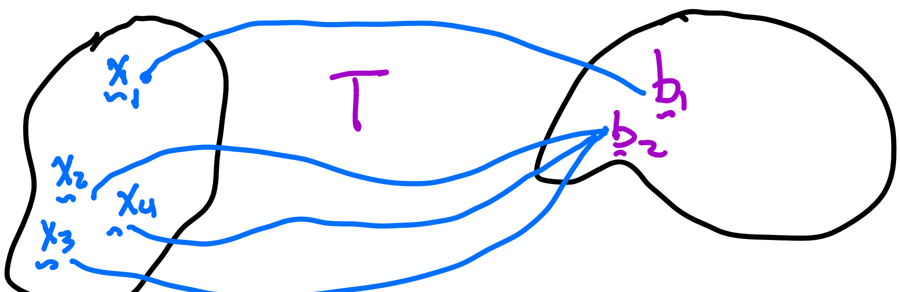
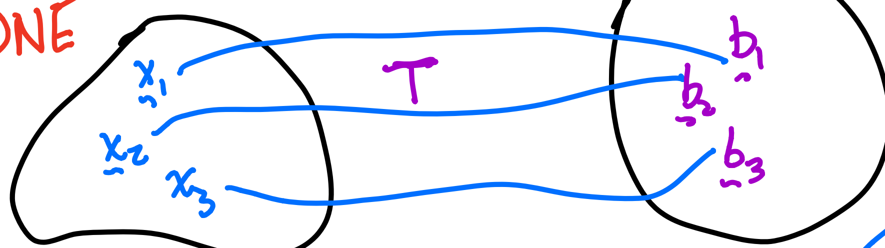
Note: The mapping T is not ONTO if $\exists\, \underset{\sim}{b} \in \mathbb{R}^m$ for which $T(\underset{\sim}{x}) = \underset{\sim}{b}$ has NO SOLUTION. //

ONTO



ONE-TO-ONE (1-1)



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Thm: $T: \mathbb{R}^n \to \mathbb{R}^m$, linear. T is 1-1 iff $T(\underset{\sim}{x}) = \emptyset$ has the trivial solution //

Thm: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear. Then

(a) T maps $\mathbb{R}^n$ ONTO $\mathbb{R}^m$ iff the columns of T span $\mathbb{R}^m$.

(b) T is 1-1 iff columns of T are LI //

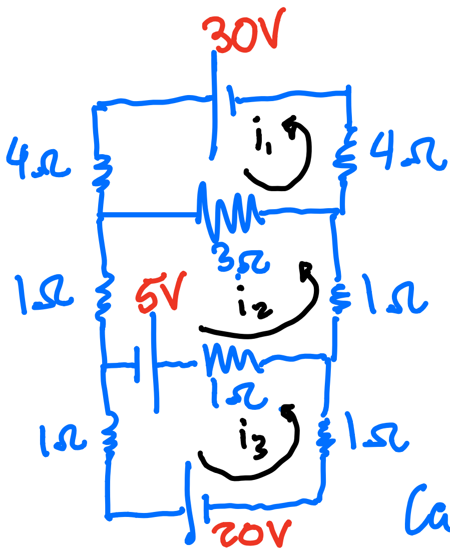
You should see how the row echelon procedure proceeds in all of the above 3 cases.

TWO APPLICATIONS OF $A\underset{\sim}{x} = \underset{\sim}{b}$:

ELECTRIC CIRCUITS

Kirchhoff's Law is (Ohm's Law)

Conservation of energy: $V = IR$.



$$4i_1 + 4i_1 + 3(i_1 - i_2) = 30$$

$$1i_2 + 3(i_2 - i_1) + 1i_2 + 1(i_2 - i_3) = 5$$

$$i_3 1 + 1(i_3 - i_2) + 1i_3 = -5 - 20$$

Can be written as $RI = V$

where $\underline{I} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}$ $\quad R = \begin{pmatrix} 11 & -3 & 0 \\ -3 & 6 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

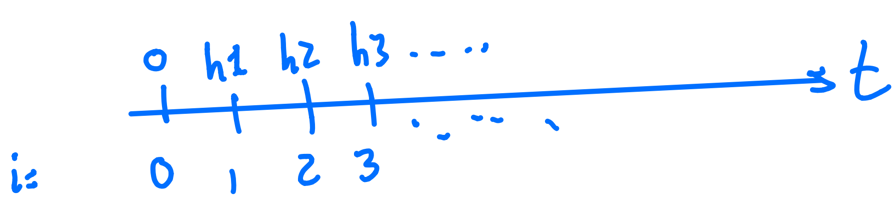
$V = \begin{pmatrix} 30 \\ 5 \\ -25 \end{pmatrix}$. The unknown currents Is are then found via Gaussian Elimination. //

SOLVING DIFFERENTIAL EQUATIONS (APPROXIMATELY):

Let $\underset{\sim}{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Find an approximation to $\underset{\sim}{x}$, which satisfies

$$\text{IVP} \begin{cases} \dfrac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x}, \; t > 0 & \text{(ODE)} \\[2ex] \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \text{ (known)} & \text{(IC)} \end{cases}$$

Discretize in time: $t_{i+1} - t_i = h$



$t_i = ih, \; i = 0, 1, 2, \ldots$

$$\frac{d\underset{\sim}{x}}{dt} \approx \frac{x(t_{i+1}) - x(t_i)}{h}$$

(*) is approximately

$$\frac{\underset{\sim}{x}(t_{i+1}) - \underset{\sim}{x}(t_i)}{h} = A\underset{\sim}{x}(t_i)$$

or $\underset{\sim}{x}(t_{i+1}) = \underset{\sim}{x}(t_i) + hA\underset{\sim}{x}(t_i)$