

Systems of Linear Equations

Simplest example: familiar (scalar = 1 Dimension):

$$\boxed{Ax=b}, \text{ another example:}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \end{cases} \quad (\star)$$

($\star$) is a system of 2 equations, with 3 unknowns, the unknowns are x_1, x_2, x_3 .

a_{ij}, b_i are known constants.

Examples of non-linear equations:

① $ax^1 = b x^{1/2} + c$ ← not power 1 in the unknown.

here, $x \in \mathbb{R}^1$, a, b, c are constants //

② $y_{n+1} = y_n + \sin(y_n)$, where $y_n \in \mathbb{R}^1$

$\sin(y_n)$ is a nonlinear function of the unknown $y_n, y_{n+1}, \dots$

here, n is an index, taking values
 $n = 0, 1, \dots$

(3) $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{23}x_3 = b_2$

The unknown (x_1, x_2, x_3) appears as a product.

The general form of a linear system
of equations:

$\underline{A}\underline{x} = \underline{b}$ e.g. see (★)

matrix $A = \{a_{ij}\}$ is known, a_{ij} are constants
row column

$1 \leq i \leq m, 1 \leq j \leq n$

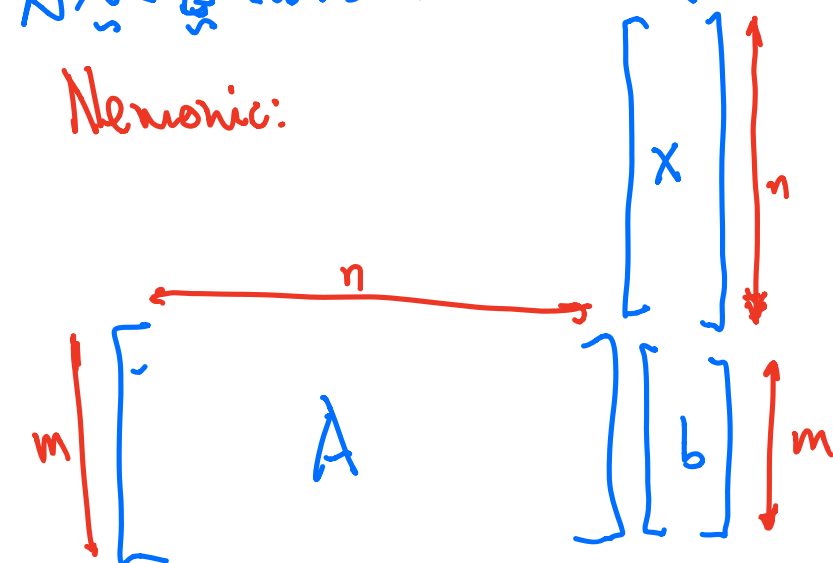
where the vectors $\underline{x} = \{x_j\}_{j=1}^n$

$\underline{b} = \{b_i\}_{i=1}^m$

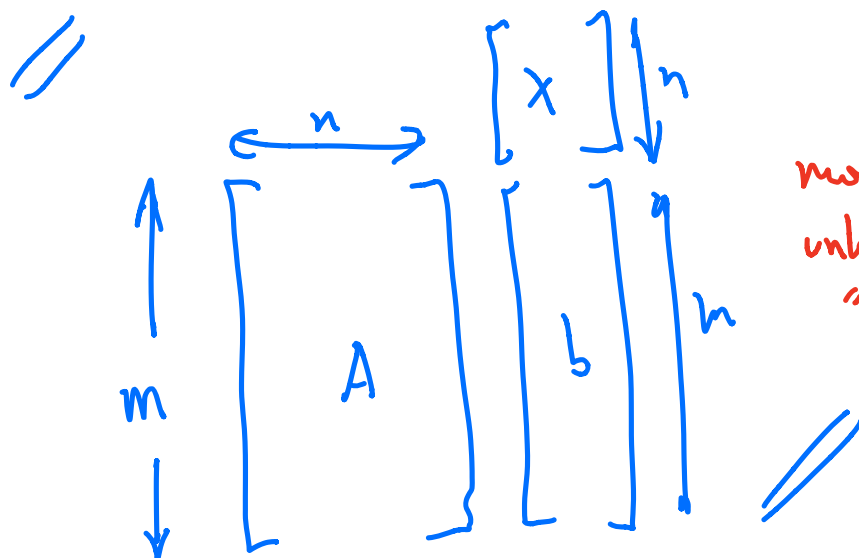
if $m=n$ A is a "square" matrix
 otherwise A is a rectangular matrix

$A\vec{x} = \vec{b}$ can be written as $\sum_{j=1}^n a_{ij}x_j = b_i \quad i=1,2,\dots,m.$

Nemonic:



more unknowns
 than equations
 "underdetermined"



more equations than
 unknowns
 "overdetermined"

Consider $\underline{A}\underline{x} = \underline{b}$

assume $\underline{A} \in \mathbb{R}^{m \times n}$ $\underline{b} \in \mathbb{R}^m$
 $\underline{x} \in \mathbb{R}^n$

Also assume $\underline{b}$, $\underline{A}$ are known.

Goal: Find $\underline{x}$. But we first ask,

is a solution $\underline{x}$ $\left\{ \begin{array}{l} \text{consistent} \\ \text{inconsistent} \end{array} \right. ?$

if consistent, is there 1 solution (unique)
or more than 1 solution?

Consistent solutions make logical sense, inconsistent ones don't.

if a, x, b are all $\mathbb{R}^1$, the equation

$ax = b$, has a consistent and unique
solution $x = b/a$ (provided $a \neq 0$).

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