FINAL EXAM TOPICS/SECTIONS

II 2.2 - 2.6

III 3.1 - 3.8

VI 6.1 - 6.6

CHAPTER II 1st order ODE

Seperable
Homogeneous
Exact

Modeling { population, tank, Cooling, Gravity/Drag }
Existence & Uniqueness
Classification

Chaper III 2nd order ODE's

C.C.
E.E.

Homogeneous/Non-homogeneous
Existence & Uniqueness (Wronskian)

MUC
Variation of Parameters
Reduction of order

Mechanical / Electrical Vibrations

Chaper VI

Laplace
Inverse

Impulse functions
Step functions
{ Convolution, Integral Equations }

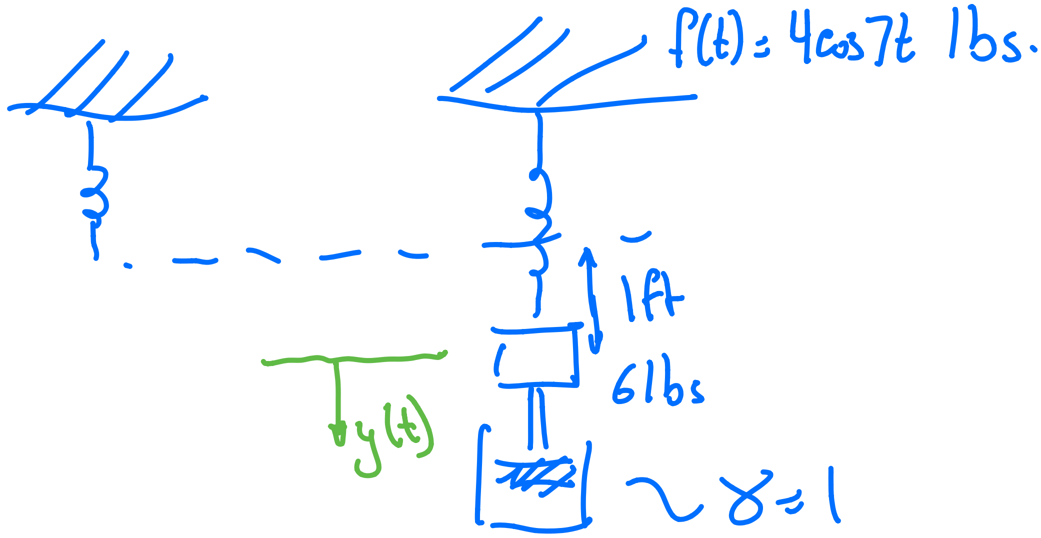
IVP

Forced/Damped Oscillator:

Damped $\gamma = 1$ Spring mass system with mass 6 lbs.

The mass travels 1 ft when the spring is loaded.

The mass is suddenly set in motion at $t=0$ by an external force of $4\cos 7t$ lb. Determine $y(t)$. The mass starts at $y(0) = -\frac{1}{2}$ ft & $y'(0) = 0$



① Find spring constant:

$$mg = k\ell$$

$$6\,\text{lb} = k \cdot 1\,\text{ft} \qquad k = 6\,\text{lb/ft}$$

$$my'' = -\gamma y' - ky + f(t)$$

$$y'' + \frac{\gamma}{m} y' + \frac{k}{m} y = \frac{f(t)}{m}$$

$$m \text{ (slugs)} = \frac{6 \text{ lb}}{32 \text{ ft}/s^2} = m$$

$$y'' + \frac{1}{m} y' + \omega^2 y = F(t) \quad \text{ODE}$$

$$\omega^2 \equiv \frac{k}{m} = \frac{6}{6/32} = 32$$

let $\delta = \frac{1}{m}$ is the damping $= \frac{32}{6}$

$$F(t) = \frac{1}{m} 4\cos 7t = \frac{64}{3} \cos 7t = \beta \cos 7t \quad t$$

I.C. $y(0) = -\frac{1}{2}$ $\quad y'(0) = 0$:

General solution:

$$y(t) = y_H(t) + y_p(t)$$

$$y_H'' + \delta y_H' + \omega^2 y_H = 0$$

$$y_H = e^{\alpha t}$$

$$\alpha^2 + \delta\alpha + \omega^2 = 0$$

$$c_1 e^{i\alpha_1 t} + c_2 e^{-i\alpha_2 t} \qquad \alpha_{1,2} = -\frac{\delta}{2} \pm \frac{1}{2}\sqrt{\delta^2 - 4\omega^2}$$

$$= -\frac{\delta}{2} \pm i\omega_d \qquad \omega_d = \frac{1}{2}\sqrt{4\omega^2 - \delta^2}$$

$$y_H = (c_1 \cos\omega_d t + c_2 \sin\omega_d t)\, e^{-\frac{\delta}{2}t}$$

$$y_p'' + \delta y_p' + \omega^2 y_p = \beta\cos(7t)$$

let $\quad y_p = A\cos(7t) + B\sin(7t)$

$$y_p' = -7A\sin(7t) + B7\cos(7t)$$

$$y_p'' = -7^2 A\cos(7t) - 7^2 B\sin(7t)$$

$$y_p'' = -7^2 y_p$$

$$-\gamma^2 y_p + \delta\left[-\gamma A\sin\gamma t + B\gamma\cos\gamma t\right] + \omega^2 y_p = \beta\cos(\gamma t)$$

$$(\omega^2-\gamma^2)\,y_p + \delta\gamma\left[-A\sin\gamma t + B\cos\gamma t\right] = \beta\cos(\gamma t)$$

Match $\cos(\gamma t)$:

$$(\omega^2-\gamma^2)A + \delta\gamma B = \beta$$

$$(\omega^2-\gamma^2)B - \delta\gamma A = 0$$

Transient Portion
(dies off as $t\to\infty$)

$$y = (c_1\cos\omega_d t + c_2\sin\omega_d t)\,e^{-\frac{\delta}{2}t} + A\cos\gamma t + B\sin\gamma t$$

Apply I.C.

$$y(0) = c_1 + A = -\frac{1}{2} \Rightarrow c_1 = -A - \frac{1}{2}$$

$y'(0)$ determines c_2.

Consider

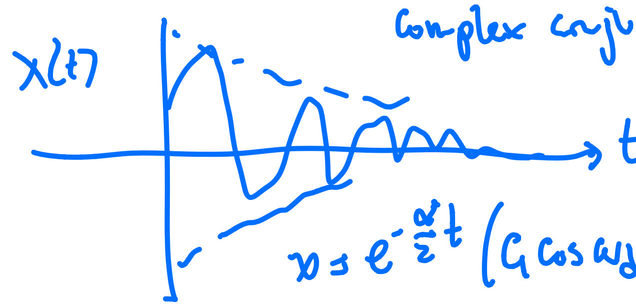
$$x'' + \alpha x' + \omega^2 x = 0$$

$$x = c_1 e^{\theta_1 t} + c_2 e^{\theta_2 t}$$

$$\theta_{1,2} = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\omega^2}$$

if $4\omega^2 > \alpha^2$ underdamped

complex conjugate roots θ_1, θ_2



$$x = e^{-\frac{\alpha}{2}t}\left(c_1 \cos \omega_d t + c_2 \sin \omega_d t\right)$$

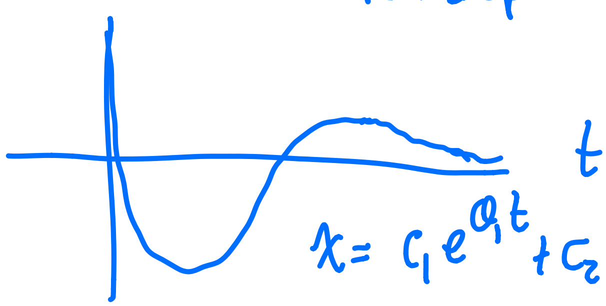
if $4\omega^2 < \alpha^2$ overdamped

θ_1, θ_2 are real & unequal

$$x = c_1 e^{\theta_1 t} + c_2 e^{\theta_2 t}$$

if $4\omega^2 = \alpha^2$ critically damped

real & equal



$$x = c_1 e^{\theta_1 t} + c_2 t e^{\theta_1 t}$$

E.E.

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0$$

$$m^2 - m + \frac{\beta}{\alpha} m + \frac{\gamma}{\alpha} = 0$$

$y = x^m$ $\qquad m^2 + \left(\frac{\beta-\alpha}{\alpha}\right) m + \frac{\gamma}{\alpha} = 0$

$\alpha (m-1) m + \beta m + \gamma = 0$ roots $m_{1,2} = -\frac{1}{2\alpha}(\beta-\alpha) \pm \frac{1}{2\alpha}\sqrt{(\beta-\alpha)^2 - 4\alpha\gamma}$

if roots are c.c.

$$y = c_1 \cos(\omega_d \ln x) + c_2 \sin(\omega_d \ln x)] x^{-c}$$

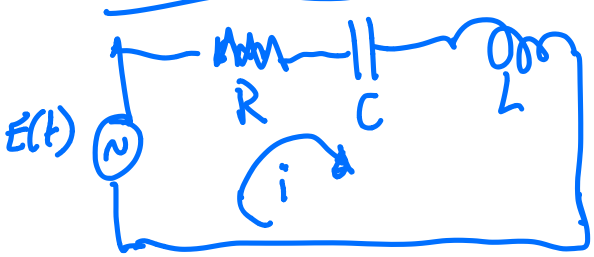
if roots are distinct & unequal

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

if roots are equal

$$y = c_1 x^{m} + c_2 \ln x \, x^{m}$$

Electrical Circuit (LRC)



$E(t) = iR + \frac{Q}{C} + L\frac{di}{dt}$ $\qquad i = \frac{dQ}{dt}$

$E(t) = \frac{dQ}{dt} R + \frac{Q}{C} + L\frac{d^2Q}{dt^2}$ $\qquad i = \frac{dQ}{dt}$

$$E'(t) = \frac{di}{dt}R + \frac{1}{C}i + L\frac{d^2i}{dt^2}$$

I.C. $Q(0)$; $Q'(0) = i(0)$ are given.

Laplace Transform:

$$\mathcal{L}(\sin \omega t) = \mathcal{L}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)$$

$$\frac{1}{2i}\mathcal{L}\left(e^{i\omega t}\right) - \frac{1}{2i}\mathcal{L}\left(e^{-i\omega t}\right)$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\left(e^{\pm i\omega t}\right) = \int_0^\infty e^{-st} e^{\pm i\omega t}\,dt$$

$$= \int_0^\infty e^{-(s \mp i\omega)t}\,dt = \frac{1}{-(s \mp i\omega)} e^{-(s \mp i\omega)t}\Big|_0^\infty$$

$$= \frac{1}{(s \mp i\omega)}$$

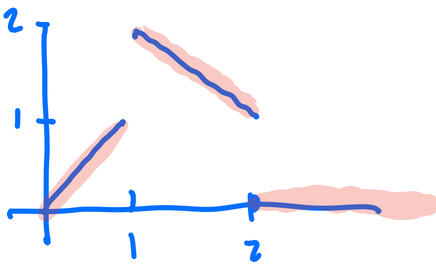
$$\mathcal{L}(\sin \omega t) = \frac{1}{2i}\left[\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right]$$

$$= \frac{1}{2i}\,\frac{s+i\omega - s + i\omega}{s^2+\omega^2} = \frac{1}{2i}\,\frac{2i\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2} \;//$$

Find $\mathcal{L}(f(t))$

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 3-t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

$f(t)$



$$f(t) = [\mu_0(t) - \mu_1(t)]\,t$$
$$+ [\mu_1(t) - \mu_2(t)]\,(3-t)$$

$$= \mu_0(t)t - \mu_1(t)t + \ldots$$

$$= \mu_0(t)t - \mu_1(t)(t-1) - \mu_1(t) +$$
$$3\mu_1(t) - 3\mu_2(t)$$
$$- \mu_1(t)t + \mu_2(t)t$$

$$\mu_0(t)t - \mu_1(t)(t-1) - \mu_1(t) +$$
$$3\mu_1(t) - 3\mu_2(t) - \mu_1(t)(t-1) - \mu_1(t)$$
$$+ \mu_2(t)(t-2) + 2\mu_2(t)$$

Convolution Theorem

$$\int_0^t f(t-\tau)\,g(\tau)\,d\tau = \mathcal{L}^{-1}\left(F(s)\,G(s)\right)$$

Integral Equation

Find solution of $\varphi(t)$

$$\mathcal{L}\left\{ \varphi(t) + \int_0^t (t-\xi)\,\varphi(\xi)\,d\xi = 1 \right\}$$

let $\Phi(s) = \mathcal{L}(\varphi)$

$$F(s) = \mathcal{L}(t) = \frac{2}{s^2}$$

$$\Phi(s) + F(s)\,\Phi(s) = \frac{1}{s}$$

$$\Phi(s)\left[1 + \frac{2}{s^2}\right] = \frac{1}{s}$$

$$\Phi(s) = \frac{1}{s}\,\frac{1}{1+2/s^2} = \frac{1}{s}\,\frac{s^2}{s^2+2}$$

$$\mathcal{L}^{-1}\left(\Phi(s)\right) = \varphi(t)$$