Math 256, Spring 2006
Scarborough
Final Exam

NAME (print) ______KEY______________________________

NAME (sign) ____

Student Number ______________________________________

Class Time (circle one) 10:00 2:00

This exam is closed book. Calculators are not allowed.
An 8.5 inch by 11 inch sheet of notes is allowed – one side of which is to be a table of Laplace Transforms.

This exam has 10 pages with 28 problems. You should scan all the problems before you begin the test so you can plan out your time. The time limit on this exam is 1 hour 50 minutes. Remember multiple choice does not necessarily mean quick answer.

Please record your answers to problems 1 through 24 on the provided Scantron sheet using a **#2 pencil**. Do all work on the exam sheet. **No scratch paper is allowed.** Also record your answer on these pages by circling the answer.

Problems 1 through 10 are worth 1 point each.
Problems 11 through 24 are worth 5 points each.
Problems 25 through 28 are worth 5 points each.
Partial credit guidelines for 25 through 28
- 5 pts – totally correct solution
- 4 pts – a very minor error that I expect that you would catch and correct in a less stressful situation
- 2 pts – procedure is basically correct with a major error or errors.

The differential equation $y' = y(1-y)$

1. (**A**) is linear (**B**) is non linear

2. (**A**) is separable (**B**) is not separable

3. (**A**) is autonomous (**B**) is not autonomous

The differential equation $t^2 + 3y - 6y' + y'' = 0$

4. (**A**) is linear (**B**) is non linear

5. (**A**) is homogeneous (**B**) is nonhomogeneous

6. (**A**) has constant coefficients (**B**) does not have constant coefficients

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The differential equation $(1-t)y'' + t^2 y' = 3y$

7. (**A**) is linear (**B**) is non linear

8. (**A**) is homogeneous (**B**) is nonhomogeneous

9, The integral $\int_0^\infty e^{-t} t^4 dt$ is

(**A**) $\mathcal{L}\left(t^4\right)$ (**B**) $\Gamma(5)$

10, The integral $\Gamma(5)$ equals

(**A**) 720 (**B**) 120 (**C**) 24

11. An integrating factor for $ty' - 3y = t^6$ is

(**A**) e^{3t} (**B**) e^{-3t} (**C**) t^3 (**D**) t^{-3}

(**E**) t^6

12. For what value of p is y^p an integrating factor for $6xy + \left(3y^2 + 9x^2\right)\dfrac{dy}{dx} = 0$?

(**A**) 3   (**B**) 2   (**C**) 1   (**D**) 0

(**E**) None of these values will work.

13. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings. Suppose that the surrounding temperature is 25° C and that the rate constant is k. Find a differential equation for the temperature $T(t)$ of the object in $^\circ$C at time t.

(**A**) $\dfrac{dT}{dt} = 25T - k$ (**B**) $\dfrac{dT}{dt} = kT - 25$

(**C**) $\dfrac{dT}{dt} = kT - 25T^2$ (**D**) $\dfrac{dT}{dt} = \dfrac{kT}{25}$

(**E**) $\dfrac{dT}{dt} = k(T - 25)$

14. For what value of k will the differential equation $y'' + 16y = \sin(kt)$ exhibit resonance?

(**A**) 0   (**B**) 1   (**C**) 4   (**D**) 8

(**E**) 16

15. Consider the differential equation $\dfrac{dy}{dt} = (1-y)(2-y)$

(**A**) The equilibrium solution $y = 1$ is asymptotically stable
and the equilibrium solution $y = 2$ is asymptotically stable

(**B**) The equilibrium solution $y = 1$ is asymptotically stable
and the equilibrium solution $y = 2$ is unstable

(**C**) The equilibrium solution $y = 1$ is unstable
and the equilibrium solution $y = 2$ is asymptotically stable

(**D**) The equilibrium solution $y = 1$ is unstable
and the equilibrium solution $y = 2$ is unstable

(**E**) $y = 1$ and $y = 2$ are not solutions to this differential equation.

16. Find the general solution to $y'' - 7y' + 12y = 0$

(**A**) $y = Ae^{-t} + Be^{-6t}$

(**B**) $y = Ae^{t} + Be^{6t}$

(**C**) $y = Ae^{-3t} + Be^{-4t}$

(**D**) $y = Ae^{3t} + Be^{4t}$

(**E**) $y = Ae^{t} + Be^{12t}$

17. Find the general solution to $y'' + 2y' + y = e^{2t}$

(**A**) $y = Ae^{-t} + Bte^{-t} + \dfrac{e^{2t}}{9}$

(**B**) $y = Ae^{-t} + Bte^{-t} + \dfrac{te^{2t}}{4} + \dfrac{e^{2t}}{2}$

(**C**) $y = Ae^{-t} + Bte^{-t} + \dfrac{e^{2t}}{4}$

(**D**) $y = Ae^{-t} + Bte^{-t} + \dfrac{te^{2t}}{2} - \dfrac{e^{2t}}{4}$

(**E**) $y = Ae^{-t} + Bte^{-t} + \dfrac{e^{2t}}{6}$

18. Find the general solution to $t^2 y'' - 7t y' + 12y = 0$

(**A**) $y = At^3 + Bt^4$

(**B**) $y = At^{-3} + Bt^{-4}$

(**C**) $y = At^2 + Bt^6$

(**D**) $y = At^{-2} + Bt^{-6}$

(**E**) $y = At + Bt^6$

19. Find $\mathcal{L}\left(u_2(t)\, t^2\right)$

(**A**) $e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

(**B**) $e^{-2s}\left[\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}\right]$

(**C**) $\frac{2e^{-2s}}{s^3}$

(**D**) $\frac{e^{-2s}}{s^2}$

(**E**) $4e^{-2s}$

20. Find $\mathcal{L}\left(\delta(t-2)\, t^2\right)$

(**A**) $e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

(**B**) $e^{-2s}\left[\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}\right]$

(**C**) $\frac{2e^{-2s}}{s^3}$

(**D**) $\frac{e^{-2s}}{s^2}$

(**E**) $4e^{-2s}$

21. What is the **best** form to use for the particular solution for the given differential equation if the method of undetermined coefficients it to be used?

$$y'' - 4y' + 4y = te^{2t}$$

Note: **best** means that all required terms are present and no unnecessary term are listed.

(A) $y_p = Ate^{2t} + Be^{2t}$

(B) $y_p = At^3e^{2t}$

(C) $y_p = At^3e^{2t} + Bt^2e^{2t}$

(D) $y_p = At^3e^{2t} + Bt^2e^{2t} + Cte^{2t} + De^{2t}$

(E) None of these are appropriate as it is not appropriate to use the method of undetermined coefficients for this problem.

22. If $f(t) = e^{2t}$ and $g(t) = \sin(3t)$ then find $\mathcal{L}(f * g(t))$.

(A) $\dfrac{3}{(s-2)^2 + 9}$

(B) $\dfrac{3}{(s-2)(s^2 + 9)}$

(C) $\dfrac{3}{s^2 + s - 7}$

(D) $\dfrac{1}{s-2} + \dfrac{3}{s^2 + 9}$

(E) $\dfrac{1}{s-2} - \dfrac{3}{s^2 + 9}$

23. If $y'' - 2y' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$ where $g(t) = \begin{cases} 1, & 0 \le t < 3 \\ 0, & 3 \le t \end{cases}$ then find $Y(s) = \mathcal{L}(y(t))$

(A) $\dfrac{1}{s^2 - 2s + 1} + \dfrac{1 - e^{-3s}}{s(s^2 - 2s + 1)}$

(B) $\dfrac{s}{s^2 - 2s + 1} + \dfrac{1 - e^{-3s}}{s(s^2 - 2s + 1)}$

(C) $\dfrac{1}{s^2 - 2s + 1} + \dfrac{e^{-3s}}{s(s^2 - 2s + 1)}$

(D) $\dfrac{s}{s^2 - 2s + 1} + \dfrac{e^{-3s}}{s(s^2 - 2s + 1)}$

(E) $\dfrac{1}{s} + \dfrac{e^{-3s}}{(s^2 - 2s + 1)}$

24. Find $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+6s+13}\right)$

(**A**) $e^{-3t}\cos(2t)$

(**B**) $e^{-3t}\left(\cos(2t)-3\sin(2t)\right)$

(**C**) $\dfrac{2}{3}e^{5t}-\dfrac{4}{3}e^{t}$

(**D**) $e^{-3t}\left(\cos(2t)-\dfrac{3}{2}\sin(2t)\right)$

(**E**) $\dfrac{1}{3}e^{5t}-\dfrac{5}{3}e^{t}$

FOR FULL CREDIT SHOW ALL WORK **Circle your final answer**

25. Solve $(2x-y)+(2y-x)\dfrac{dy}{dx}=0,\ \ y(1)=3$

$$\frac{\partial\varphi}{\partial x}=2x-y$$
$$\varphi=x^2-xy+h(y)$$
$$\frac{\partial\varphi}{\partial y}=-x+h'(y)=2y-x$$
$$h'(y)=2y$$
$$h(y)=y^2$$
$$\varphi(x,y)=C \quad\Rightarrow\quad x^2-xy+y^2=C$$
$$1^2-(1)(3)+3^2=C \quad\Rightarrow\quad C=7$$

$$x^2-xy+y^2=7$$

FOR FULL CREDIT SHOW ALL WORK **Circle your final answer**

26. One solution to $t\ y'' - 2(t+1)y' + (t+2)y = 0$ is $y_1 = e^t$

(a) Verify that $y_1 = e^t$ is a solution to this differential equation.

$y_1 = e^t$, $y_1' = e^t$, $y_2'' = e^t$

$ty'' - 2(t+1)y' + (t+2)y = te^t - 2(t+1)e^t + (t+2)e^t = 0$

(b) Find a second linearly independent solution to this differential equation.

$$p(t) = \frac{-2(t+1)}{t} = -2 - \frac{2}{t}$$

$$y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2} dt = e^t \int \frac{e^{\int 2+\frac{2}{t}dt}}{e^{2t}} dt = e^t \int \frac{e^{2t+2\ln t}}{e^{2t}} dt = e^t \int \frac{t^2 e^{2t}}{e^{2t}} dt = e^t \int t^2 dt = \frac{1}{3} t^3 e^t$$

Since any consatant multiple of a solution is also a solution we will use

$y_2 = t^3 e^t$

FOR FULL CREDIT SHOW ALL WORK **Circle your final answer**

27. Find the general solution to $y'' + 9y = \sec(3t)$

$$y_1 = \cos(3t),\;\; y_2 = \sin(3t),\;\; W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^3(3t) = 3(\cos^2(3t) + \sin^3(3t)) = 3$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1{}' = \frac{1}{3}\sec(3t)\begin{vmatrix} 0 & \sin(3t) \\ 1 & 3\cos(3t) \end{vmatrix} = -\frac{1}{3}\sec(3t)\sin(3t) = -\frac{1}{3}\tan(3t)$$

$$u_1 = \int -\frac{1}{3}\tan(3t)dt = -\frac{1}{9}\ln\left|\sec(3t)\right|$$

$$u_2{}' = \frac{1}{3}\sec(3t)\begin{vmatrix} \cos(3t) & 0 \\ -3\sin(3t) & 1 \end{vmatrix} = \frac{1}{3}\sec(3t)\cos(3t) = \frac{1}{3}$$

$$u_1 = \int \frac{1}{3}dt = \frac{t}{3}$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{9}\cos(3t)\ln\left|\sec(3t)\right| + \frac{t}{3}\sin(3t)$$

$$y = A\cos(3t) + B\sin(3t) - \frac{1}{9}\cos(3t)\ln\left|\sec(3t)\right| + \frac{t}{3}\sin(3t)$$

FOR FULL CREDIT SHOW ALL WORK **Circle your final answer**

28. Solve $y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0$

$\mathcal{L}(y(t)) = Y(s)$

$\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s) - 1$

$\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$

$s^2Y - s + 2(sY - 1) + 2Y = e^{-\pi s}$

$$Y = \frac{s+2}{s^2+2s+2} + \frac{e^{-\pi s}}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} + e^{-\pi s}\frac{1}{(s+1)^2+1}$$

$y = e^{-t}(\cos t + \sin t) + u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$

$y = y = e^{-t}(\cos t + \sin t) - u_\pi(t)e^{-(t-\pi)}\sin t$