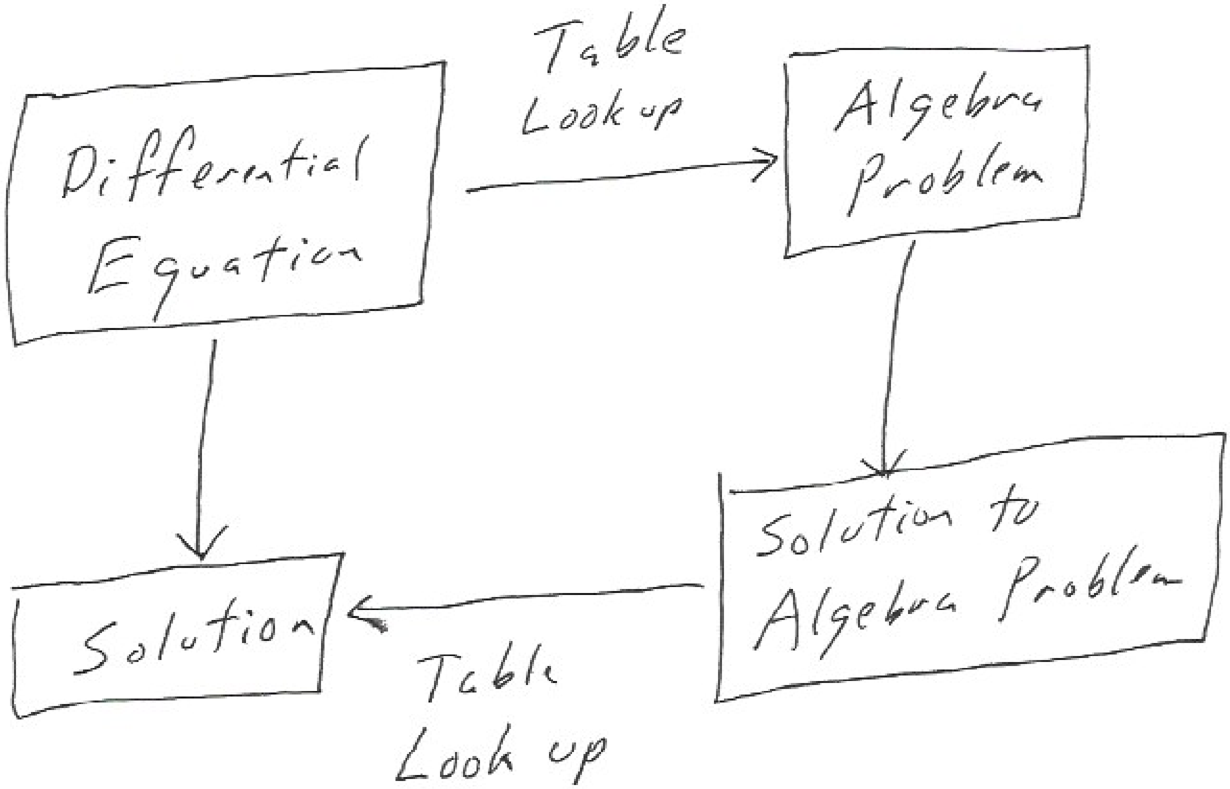
Math 256

Introduction to Laplace Transforms

Lecture 19

An application of Laplace Transforms



Improper Integrals

If $f(t)$ is defined for $t \geq 0$ the improper integral $\int_0^\infty f(t)\,dt$ is defined as $\lim\limits_{b\to\infty} \int_0^b f(t)\,dt$

If this limit exists the integral converges, otherwise it diverges

Example: If $s > 0$

$$\int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left. \frac{-e^{-st}}{s} \right|_0^b$$

$$= \lim_{b\to\infty} -\frac{e^{-sb}}{s} + \frac{1}{s}$$

$$= \frac{1}{s}$$

Integral Transforms

A definite integral

$$\int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt = F(s)$$

that transforms a function $f(t)$ into a function of s, $F(s)$ is called an integral transform. $K(s,t)$ is called the kernel of the transform

Examples: Fourier Transform
Radon Transform
Z Transform
Laplace Transform.

All integral transforms are

Linear

$$F(s) = \int_\alpha^\beta K(s,t) f(t)\, dt$$

$$G(s) = \int_\alpha^\beta K(s,t) g(t)\, dt$$

$$\int_\alpha^\beta K(s,t)[f(t) + g(t)]\, dt$$

$$= \int_\alpha^\beta K(s,t) f(t)\, dt + \int_\alpha^\beta K(s,t) g(t)\, dt$$

$$= F(s) + G(s)$$

and if k is a constant

$$\int_\alpha^\beta K(s,t)\, k f(t)\, dt$$

$$= k \int_\alpha^\beta K(s,t) f(t)\, dt$$

$$= k F(s$$

Definition Let f be a function defined for $t > 0$, then the integral transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

is called the Laplace Transform of f, provided that this integral converges.

Examples:

(1) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \frac{1}{s}, \quad s > 0$

(2) $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt$

$$= \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\,dt$$

$$= \lim_{b\to\infty} \left[-\frac{e^{-(s-a)b}}{s-a} + \frac{1}{s-a} \right]$$

$$= \frac{1}{s-a}, \quad s > a.$$

For what functions are we guaranteed to have a Laplace Transform?

Definition A function f is piecewise continuous if on every finite interval $[a, b]$ f has at most a finite number of jump or removable discontinuities and no infinite discontinuities

Examples:

<u>Definition</u> A function f is of <u>exponential order</u> if there are constants, K, a, T where $|f(t)| \leq Ke^{at}$ for all $t \geq T$

<u>Examples:</u>

Theorem If f is piecewise continuous on $[0, \infty)$ and is of exponential order with $|f(t)| \le K e^{at}$ then

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

exists for $s > a$.

Notation Inverse Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\}$$

$$f(t) - \mathcal{L}^{-1}\{F(s)\}$$

Examples: $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

Example: $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \, dt$

$$= \lim_{b\to\infty} \int_0^b t e^{-st} \, dt$$

$$u = t \qquad dv = e^{-st} \, dt$$

$$du = dt \qquad v = -\frac{1}{s} e^{-st}$$

$$= \lim_{b\to\infty} -\frac{t}{s} e^{-st} \Big|_0^b + \frac{1}{s} \int_0^b e^{-st} \, dt$$

$$= \lim_{b\to\infty} \frac{-e^{-st}}{s^2} \Big|_0^b = \frac{1}{s^2}$$

Example: $\mathcal{L}\{t^2\}$

$p \geq 0$ for now

Find

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}\, t^p \, dt$$

Let $x = st \Rightarrow t = \frac{x}{s}$

$dx = s\,dt \Rightarrow dt = \frac{dx}{s}$

$$= \int_0^\infty \left(\frac{x}{s}\right)^p e^{-x} \frac{dx}{s}$$

$$\mathcal{L}\{t^p\} = \frac{\int_0^\infty x^p e^{-x}\, dx}{s^{p+1}}$$

<u>Definition</u> Define the Gamma function

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\, dx$$

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx$$

$$\Gamma(1) = \Gamma(0+1) = \int_0^\infty x^0 e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^\infty = 1$$

$$\boxed{\Gamma(1) = 1}$$

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx \quad , \quad p > 0$$

$$u = x^p \qquad dv = e^{-x}\,dx$$

$$du = p x^{p-1}\,dx \qquad v = -e^{-x}$$

$$= -x^p e^{-x}\Big|_0^\infty + p\int_0^\infty x^{p-1} e^{-x}\,dx$$

$$\Gamma(p+1) = p\,\Gamma(p) \quad , \quad p > 0$$

$$\Gamma(5) = 4\,\Gamma(4)$$
$$= 4 \cdot 3\,\Gamma(3)$$
$$= 4 \cdot 3 \cdot 2 \cdot \Gamma(2)$$
$$= 4 \cdot 3 \cdot 2 \cdot 1 \cdot \Gamma(1)$$
$$= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1$$

$$\Gamma(5) = 4!$$

If n is a nonnegative integer

$$\Gamma(n+1) = n!$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad p > -1, \quad s > 0$$

If n is a positive integer

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

<u>Example</u>

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\, dx$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}\, dx$$

$$= \int_0^\infty 2\, e^{-u^2}\, du$$

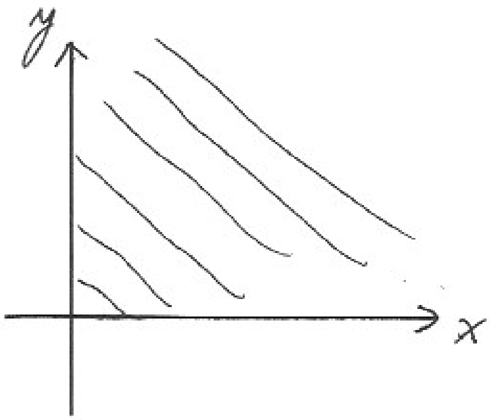
Let $I = \int_0^\infty e^{-x^2} dx$, $I = \int_0^\infty e^{-y^2} dy$

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

Convert to Polar Coordinates (15)

$$x^2 + y^2 = r^2 \quad dx\,dy = r\,dr\,d\theta$$



$$\{(x,y) \mid 0 \le x < \infty,\ 0 \le y < \infty\}$$

$$= \{(r,\theta) \mid 0 \le r < \infty,\ 0 \le \theta \le \tfrac{\pi}{2}$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty r e^{-r^2}\,dr\,d\theta$$

$$= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]\Big|_0^\infty = \frac{\pi}{4}$$

$$I^2 = \frac{\pi}{4} \implies I = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$$

So $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty 2e^{-u^2}\,du = \sqrt{\pi}$

$$\mathcal{L}\{\sqrt{t}\} = \mathcal{L}\{t^{\frac{1}{2}}\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}$$

$$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\mathcal{L}\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

OSU connection to Laplace Transforms

One of the earliest extensive tables of Laplace Transforms was developed by Fritz Oberhettinger.

Other useful functions defined by integrals

Error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

Sine integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

Beta function

$$B(n,m) = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}$$

Elementary Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s},\quad s>0$
2	e^{at}	$\frac{1}{s-a},\quad s>a$
3	$t^n,\quad n=$ positive integer	$\frac{n!}{s^{n+1}},\quad s>0$
4	$t^p,\quad p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}},\quad s>0$
5	$\sin at$	$\frac{a}{s^2+a^2},\quad s>0$
6	$\cos at$	$\frac{s}{s^2+a^2},\quad s>0$
7	$\sinh at$	$\frac{a}{s^2-a^2},\quad s>\lvert a\rvert$
8	$\cosh at$	$\frac{s}{s^2-a^2},\quad s>\lvert a\rvert$
9	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\quad s>a$
10	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\quad s>a$
11	$t^n e^{at},\quad n=$ positive integer	$\frac{n!}{(s-a)^{n+1}},\quad s>a$
12	$u_c(t)$	$\frac{e^{-cs}}{s},\quad s>0$
13	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14	$e^{ct}f(t)$	$F(s-c)$
15	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right),\quad c>0$
16	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17	$\delta(t-c)$	e^{-cs}
18	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
19	$(-t)^n f(t)$	$F^{(n)}(s)$