Does

A $\begin{cases} \dfrac{dy}{dx} = y^3 \\ y(0) = 0 \end{cases}$ have a solution? is it unique?

Consider more generally

B $\begin{cases} \dfrac{dy}{dx} = y^3 = f(x,y) \\ y(x_0) = y_0 \end{cases}$

1. we note that f is defined (and continuous) for $\forall x, y$
2. we note that $\dfrac{\partial f}{\partial y} = 3y^2$ is also defined (and continuous) for $\forall x, y$

So the existence & uniqueness theorem says there's a unique solution near (x_0, y_0).

We can easily solve B (seperable)

$$\frac{dy}{y^3} = dx \Rightarrow y^2 = -2x + c$$

Apply $y(x_0) = y_0$

$y_0^2 = -2x_0 + C \Rightarrow C = y_0^2 + 2x_0$ (C)

Now, specialize to (A):

(*) $y(0) = 0$. By (C) $C = 0$

$\therefore y^2 = -2x$ (**)

How can (*) & (**) be satisfied?

By making $y=0$ the solution.

Show that if $M(x,y)\,dx + N(x,y)\,dy = 0$

is homogeneous, it can be written as

$$\frac{dy}{dx} = G(y/x)$$

(1) if homogeneous

$$\lambda^n M(x,y)\,dx + \lambda^n N(x,y)\,dy = 0$$

$$M(\lambda x, \lambda y)\lambda\,dx + N(\lambda x, \lambda y)\lambda\,dy = 0$$

let $u = y/x$

$$M(\lambda x, \lambda x u)\lambda dx + N(\lambda x, \lambda x u)\lambda dy = 0$$

factor λx:

$$M(1,u)\lambda^n dx + N(1,u)\lambda^n dy = 0$$

or $M(u)\lambda^n dx + N(u) dy \lambda^n = 0$

$$\therefore \frac{dy}{dx} = -\frac{M(u)}{N(u)} \equiv G(u)$$

Solve

(*) $y\,dx + x(\ln x - \ln y - 1)dy = 0$

$y(1) = e$

$M dx + N dy = 0$ is it homogeneous?

let $x \to \lambda x$, $y \to \lambda y$

$M(y) = y \quad M(\lambda y) = \lambda y = \lambda M(y)$

$N(x,y) = x(\ln x/y - 1) \quad N(\lambda x, \lambda y) = \lambda x(\ln(x/y) - 1)$
$= \lambda N(x,y)$

$\therefore$ Homogeneous!

let $y = ux : dy = du\,x + u\,dx$

$(*)\quad y\,dx + x(\ln x - \ln y - 1)\,dy = 0$

$\frac{y}{x}\,dx + (\ln(x/y) - 1)\,dy = 0$

$u\,dx + (\ln(1/u) - 1)\,dy = 0$

$u\,dx - (\ln u + 1)(x\,du + u\,dx) = 0$

$-\dfrac{u\,dx}{1+\ln u} + x\,du + u\,dx = 0$

$-\dfrac{dx}{x}\left[\dfrac{\ln u}{1+\ln u}\right]u = du$

$-\dfrac{dx}{x} = \dfrac{1}{u\ln u}[1+\ln u]\,du$

$-\dfrac{dx}{x} = \dfrac{du}{u\ln u} + \dfrac{du}{u}$

let $w = \ln u \qquad dw = \frac{1}{u}du$

$-\dfrac{dx}{x} = \dfrac{dw}{w} + \dfrac{du}{u}$

$$-\ln|x| = \ln|w| + \ln|u| + c$$

$$-\ln|x| + c = \ln|\ln u| + \ln|u|$$

$$\ln|x|^{-1} + c = \ln|u \ln u|$$

$$\frac{K}{x} = u\ln u \quad \text{or} \quad \frac{K}{x} = \frac{x}{y}\ln\left(\frac{x}{y}\right)$$

$$K = \frac{x^2}{y}\ln\left(\frac{x}{y}\right)$$

Apply $y(1) = e$

$$K = \frac{1}{e}\ln\left(\frac{1}{e}\right) = \frac{1}{e}\ln e^{-1} = -\frac{1}{e}\ln e = -\frac{1}{e}$$

$$\therefore -\frac{1}{e} = \frac{x^2}{y}\ln\left(x/y\right) \quad //$$

Could also use $x = vy$ $\quad \therefore v = x/y$

$$dx = dv\,y + v\,dy$$

then

$$(*) \quad y\,dx + x(\ln x - \ln y - 1)\,dy = 0$$

$$dx + v(\ln v - 1)\,dy = 0$$

$$dv\,y + dy\,v + v(\ln v - 1)\,dy = 0$$

$$-y\,dv = v\ln v\,dy$$

$$\frac{dv}{v\ln v} = -\frac{dy}{y} \qquad \text{let } w = \ln v \quad dw = \frac{1}{v}dv$$

$$\frac{dw}{w} = -\frac{dy}{y} \Rightarrow \ln|w| = -\ln|y| + c$$

$$\ln(\ln v) + \ln|y| = c$$

$$\ln(y\ln v) = c$$

$$\ln\left(y\ln\left(x/y\right)\right) = c$$

$$y\ln\left(x/y\right) = e^c = K$$

Generally

$$y_0 \ln\left(\frac{x_0}{y_0}\right) = K$$

$$e \ln\left(\frac{1}{e}\right) = K$$

$$e \ln e^{-1} = -e \ln e = K$$

$$\therefore K = -e$$

$$y \ln\left(\frac{x}{y}\right) = -e$$

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Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

Manipulate to get

$$xy(1+xy^2) = \frac{dx}{dy}$$

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} - yx = y^3x^2 \quad (\text{A})$$

Think of $x(y)$ then $(\star)$ is a Bernoulli equation, with $n=2$

Substitute $w(y) = x(y)^{1-n} = x^{-1}$

$$\frac{dw}{dy} = -\frac{1}{x^2}\frac{dx}{dy}$$

Subtitute into $\star$:

$$-x^2\frac{dw}{dy} - yx = y^3 x^2$$

Divide by x^2:

$$-\frac{dw}{dy} - \frac{y}{x} = y^3 \quad \text{, since } x = 1/w$$

$$-\frac{dw}{dw} - yw = y^3$$ a linear ODE for $w = w(y)$

$$I = e^{-\int y\,dy} = e^{-y^2/2}$$

$$\frac{d}{dy}(wI) = -y^3 e^{-y^2/2}$$

$$w\,I = -\int^{y} s^3 e^{-s^2/2}\,ds + c$$

$$w = -e^{y^2/2}\int^{y} s^3 e^{-s^2/2}\,ds + ce^{y^2/2}$$

Find $\int s^3 e^{-s^2/2}\,ds$

$u = s^2$ $\qquad v = -e^{-s^2/2}$

$du = 2s\,ds$ $\qquad dv = se^{-s^2/2}\,ds$

$$\int s^3 e^{-s^2/2}\,ds = -s^2 e^{-s^2/2} - 2\int s e^{-s^2/2}\,ds$$

$$\left.\begin{aligned} u &= -s^2/2 \\ du &= -s\,ds \end{aligned}\right\}$$

$$\int s^3 e^{-s^2/2}\,ds = -s^2 e^{-s^2/2} + 2\int e^{u}\,du = -s^2 e^{-s^2/2} + 2e^{-s^2/2}$$

$$w = -e^{y^2/2}\left[-y^2 e^{-y^2/2} + 2e^{-y^2/2}\right] + ce^{y^2/2}$$

$$w = -y^2 - 2 + ce^{y^2/2} = \frac{1}{x}$$

$$x(y) = \frac{1}{ce^{y^2/2} - y^2 - 2}$$ //