4.1 SOLUTION TO n^{th} Order LINEAR ODE's

Notation:

Denote by $\mathcal{L}$ the Linear Differential operator

$$\boxed{\mathcal{L} \equiv \sum_{i=0}^{n} a_i \frac{d^i}{dx^i}, \quad a_i = a_i(x) \text{ (not all zero)}}$$

e.g. $n=2$, with $y = y(x)$

$$\mathcal{L} = a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0, \text{ apply to } y:$$

$$\mathcal{L}y = a_2 \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y$$

Suppose

$$a_2 = x^2 \quad a_1 = -1 \quad a_0 = \frac{1}{x^2}$$

then $\mathcal{L}y = x^2 \frac{d^2}{dx^2} y - \frac{dy}{dx} + \frac{1}{x^2} y$ //

HOMOGEOUS Linear Differential Equation of order n is:

$$\mathcal{L}y = 0$$

$\mathcal{L}$ is n^{th} order linear operator.

The Non-Homogeneous counterpart:

$$\mathcal{L}y = f(x)$$

$f(x) \neq 0$, known.

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EXISTENCE & UNIQUENESS of solutions to

$\mathcal{L}y = f(x)$. Consider case

when $\mathcal{L}$ is 2nd order;

let $y = y(t)$

$$(\star)\begin{cases} y'' + p(t)y' + q(t)y = g(t) & \text{ODE} \\ y(t_0) = y_0 \quad y'(t_0) = y_1 & \text{I.C.} \end{cases}$$

$(\star)$ is called an initial value problem (IVP)

A unique solution exists in $t \in (\alpha, \beta)$, where $t_0 \in (\alpha, \beta)$ if p, q, g are continuous in (α, β).

Match $\mathcal{L} y = f(t)$ to $(\star)$

$$\mathcal{L} y = a_2(t) y'' + a_1(t) y' + a_0(t) y = f(t)$$

divide by $a_2(t)$

$$y'' + \frac{a_1}{a_2} y' + \frac{a_0}{a_2} y = \frac{f}{a_2}$$

Compare to $(\star)$

$$p(t) = \frac{a_1(t)}{a_2(t)} \quad q(t) = \frac{a_0}{a_2}$$

$$g(t) = \frac{f(t)}{a_2(t)}$$

ex) $2\cos^2 t\, y'' + y = 5\tan t$ ODE

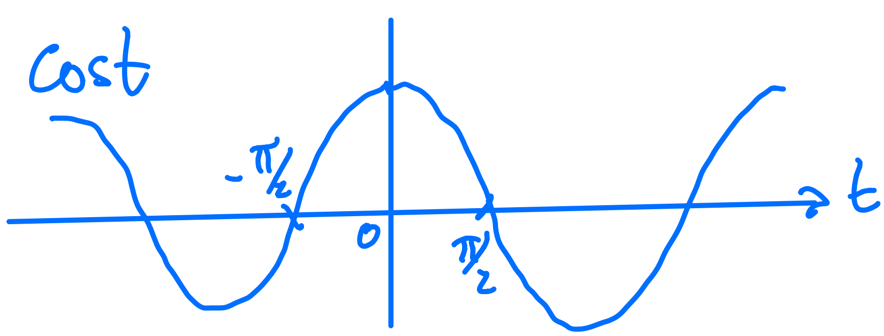
$y(0) = 0 \quad y'(0) = 1$ I.C.

For this IVP,
want to determine whether a solution exists
& is unique:

$$y'' + \frac{1}{2}\sec^2 t\, y = \frac{5}{2}\sec^2 t \tan t$$

here $p = 0 \quad q = \frac{1}{2}\sec^2 t \quad g(t) = \frac{5}{2}\sec^2 t \tan t$

$y_0 = 0 \quad y_1 = 1 \quad t_0 = 0$



when $\cos t = 0$ q, g are singular,
we need to include $t = 0$ in the interval, because
the initial condition is specified at $t = 0$.

We'll chose $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which includes $t=0$. In that range we expect a unique solution.

The actual solution to the IVP is

$$y(t) = \tan t$$

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PRINCIPLE OF SUPERPOSITION

Implies LINEARITY

We say that a system is linear if it obeys the principle of superposition:

For functions, say $f(x)$

If $f(x_1 + x_2 + \dots) = f(x_1) + f(x_2) + f(x_3) \dots$

and

$f(ax) = a\, f(x)$ (homogeneity)

Then $f(x)$ obeys the principle of

superposition. If so, $f(x)$ is linear

e.g. of a (nonlinear) process that does not obey principle of superposition:

$$f(x) = e^x$$

$$f(x_1 + x_2) = e^{(x_1 + x_2)}$$

is that the same as

$$e^{x_1} + e^{x_2} \; ?$$ NO

$$f(ax) = e^{ax}$$

is it the same as ae^x? NO.

Consider $\mathcal{L}y = 0$ (2nd order)

Suppose $\mathcal{L}y_1 = 0$

and $\mathcal{L}y_2 = 0$

y_1 & y_2 are called solutions to

$$\mathcal{L}y = 0$$

Now, consider

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}y_1 + \mathcal{L}y_2$$

$$\mathcal{L} = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

$\mathcal{L}$ is said to be a "linear operator".

The General Solution to the 2nd order Linear ODE

$$\mathcal{L}y = 0 \qquad y = y(x)$$

is $\quad y = c_1 y_1(x) + c_2 y_2(x)$

where c_1, c_2 are constants and $y_1(x)$ and $y_2(x)$ are linearly independent.

Note $\mathcal{L}[c_i y_i] = c_i \mathcal{L}y_i$

e.g. $\quad y'' + \omega^2 y = 0 = \mathcal{L}y = 0 \quad , \; y = y(x)$

$$\mathcal{L} = \frac{d^2}{dx^2} + \omega^2$$

$y_1 = c_1 \cos \omega x \qquad y_2 = c_2 \sin \omega x$

$y = c_1 \cos \omega x + c_2 \sin \omega x$

$$\mathcal{L} y = \mathcal{L}[c_1 \cos \omega x + c_2 \sin \omega x] = 0$$

$$= \mathcal{L}[c_1 \cos \omega x] + \mathcal{L}[c_2 \sin \omega x] = 0$$

$$= c_1 \underbrace{\mathcal{L} \cos \omega x}_{0} + c_2 \underbrace{\mathcal{L} \sin \omega x}_{0} = 0$$

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