Population Dynamics

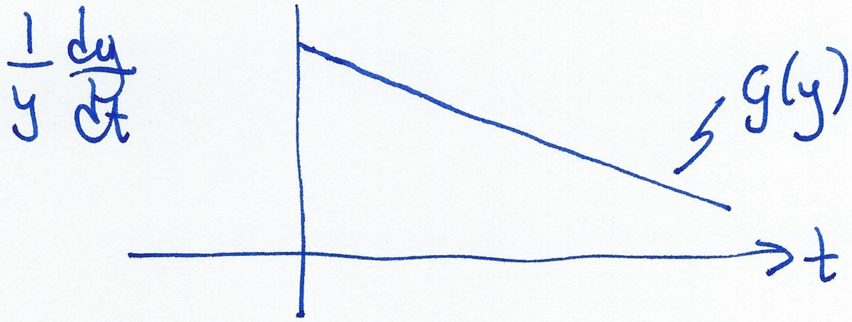
$$\frac{dy}{dt} = ry$$ the population grows/decays exponentially.

$$\frac{1}{y}\frac{dy}{dt} = r$$ the relative rate is constant.

But biological populations often do not grow or decay exponentially forever... in a closed system there's ~~mostly~~ factors limiting growth (e.g. food, space, etc)

In US early population growth rates were well modelled by $\frac{1}{y}\frac{dy}{dt} = r$. But after WWII

a better model $$\frac{1}{y}\frac{dy}{dt} = r - by = f(y)$$



$r > 0$

$b > 0$

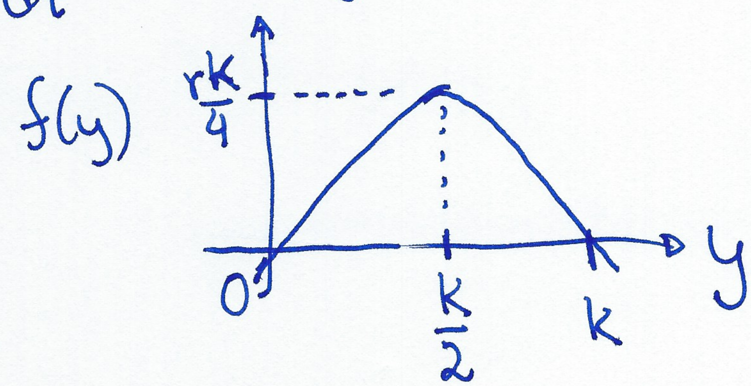
so $\frac{dy}{dt} = y\,g(y) = ry\left(1 - \frac{b}{r}y\right)$

equil pts: $y = 0$ and $y = \frac{r}{b} \equiv K$ is called the "carrying capacity"

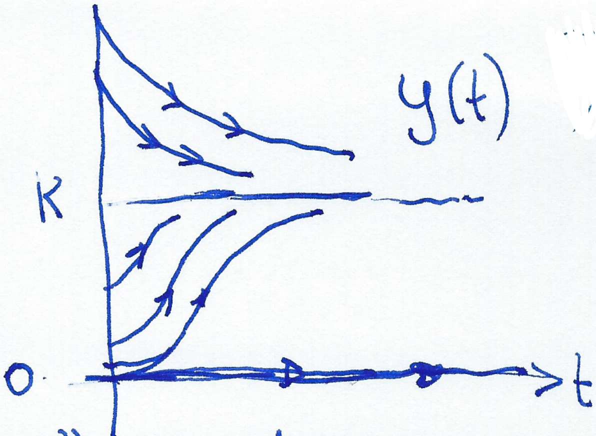
$$\frac{dy}{dt} = ry\left(1 - y/K\right)$$

"Logistic Equation"

$$\frac{dy}{dt} = f(y) = ry\left(1 - y/K\right)$$



ODE $\frac{dy}{dt} = ry(1 - y/K)$



By building a slope field ~~phase~~ plot... but we'll see above is correct by solving ODE.

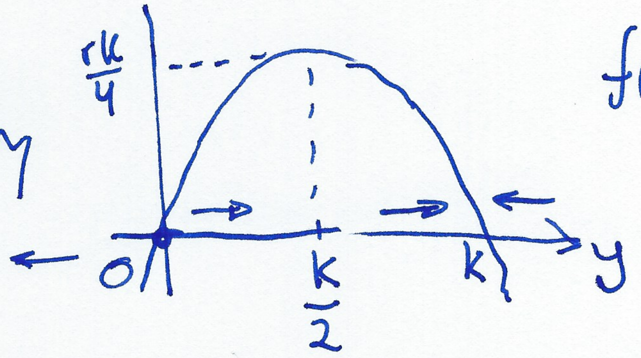
<u>Note</u>: suppose $|y| << 1$ then $ry - y^2/K \approx ry$ because $|y|^2$ is even smaller that $|y|$

so ODE is, approximately,

$$\frac{dy}{dt} \sim ry \quad \text{for small } r$$

so $y \sim y_0 e^{rt}$ for small y $\left\{ \begin{array}{l} |y_0| \text{ small} \\ \text{and} \\ t \text{ small} \end{array} \right.$

i.e. small populations grow exponentially.

Why 

$f(y) = ry(1 - y/K)$

$r > 0$

$K > 0$

$y = 0$ is unstable (slope positive) for $y \geq 0$

$y = K$ is stable (slope is negative) for $y > K$

$f(y)$ is maximum at $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial y} = r(1 - y/K) - \frac{r}{K}y = 0 \Rightarrow 1 - \frac{2}{K}y = 0$$

or $y = K/2$

$$f(K/2) = r\frac{K}{2}\left(1 - \frac{K}{2K}\right) = \frac{rK}{4}$$

//

let's find out how to solve

$$\frac{dy}{dt} = ry(1 - y/K)$$

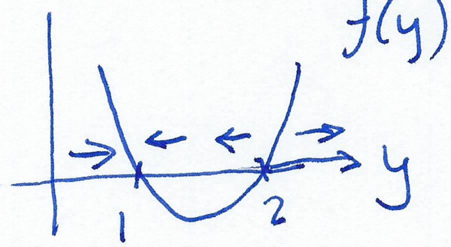
clearly, separable $\frac{dy}{y(1-y/K)} = r\,dt$

More generally, consider ODE's of the form

$$\frac{dy}{dt} = k(a-y)(b-y) = f(y)$$

e.g. logistics equation, 2nd order chemical reactions, etc

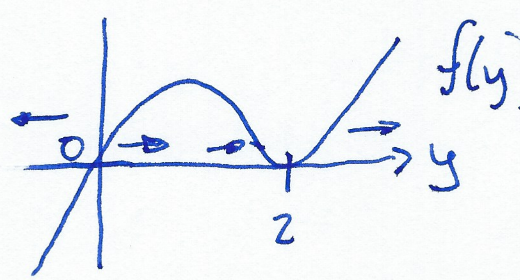
ex) $\frac{dy}{dt} = (1-y)(2-y)$



$y=1$ stable

$y=2$ unstable

ex) $\frac{dy}{dt} = y(2-y)^2$



$y=0$ unstable

$y=2$ semi-stable

Partial fractions: take $f(x) = \dfrac{g(x)}{h(x)}$

Where $h(x) = (x-a)(x-b)$, a, b are known

1st, write $f(x) = g(x) \cdot \dfrac{1}{h(x)}$ and focus on $\dfrac{1}{h(x)}$

$$\frac{1}{h(x)} = \frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{x-b}$$

Multiply both sides by $h(x)$

$$1 = A(x-b) + B(x-a)$$

Collect powers of x:

$$0x^1 + 1x^0 = (A+B)x^1 + (-Ab-Ba)x^0$$

Match coefficients of powers of x:

$$0 = A+B \rightarrow B = -A$$

$$1 = -Ab - Ba \therefore 1 = -Ab + Aa = A(a-b)$$

$$\therefore A = \frac{1}{a-b} \quad B = -A = \frac{1}{b-a}$$

$$\therefore f(x) = \frac{g(x)}{h(x)} = \frac{g(x)}{(x-[a-b])} + \frac{g(x)}{(x-[b-a])}$$

Aside: partial fractions (more general)

ex) $$\frac{1}{(x-a)(x-b)^3} = \frac{A}{x-a} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \frac{B_3}{(x-b)^3}$$

Multiply both sides by $(x-a)(x-b)^3$

$$1 = A(x-b)^3 + B_1(x-a)(x-b)^2 + B_2(x-a)(x-b) + B_3(x-a)$$

Collect coefficients of equal powers in x:

$$\ldots 0x^1 + 1x^0 = [Ab^3 - ab^2B_1 + abB_2 - aB_3]x^0$$
$$+ [A3b^2 + b^2B_1 + 2aB_1 - aB_2 - bB_2 + B_3]x^1$$
$$+ [-3bA - 2B_1 - aB_1 + B_2]x^2$$
$$+ [A + B_1]x^3$$

Match powers of x and solve for A, B_1, B_2, B_3 //

see other forms amenable to partial fraction expansions on web...

Return to

$$\frac{dy}{y(1-y/k)} = r\,dt$$

$$\frac{1}{y(1-y/k)} = \frac{A}{y} + \frac{B}{1-y/k}$$

$$1 = A(1-y/k) + By$$

$$A=1 \quad \& \quad 0 = -A/k + B \Rightarrow B = \frac{1}{k}$$

$$\int \frac{dy}{y(1-y/k)} = \int \frac{dy}{y} + \frac{1}{k}\int \frac{dy}{1-y/k} = \int r\,dt = rt + \tilde{c}$$

$$\ln y - \ln(1-y/k) = rt + \tilde{c}$$

$$\ln\left[\frac{y}{1-y/k}\right] = rt + \tilde{c}$$

$$\frac{y}{1-y/k} = Ce^{rt} \Rightarrow y = Ce^{rt} - \frac{y}{k}Ce^{rt}$$

$$y + \frac{y}{k}Ce^{rt} = Ce^{rt}$$

$$y\left(1 + \frac{C}{k}e^{rt}\right) = Ce^{rt}$$

$$y = \frac{Ce^{rt}}{1 + C/k\, e^{rt}} = \frac{e^{rt}\, C}{e^{rt}\left[e^{-rt} + C/k\right]}$$

$$y = \frac{CK}{C + Ke^{-rt}}$$

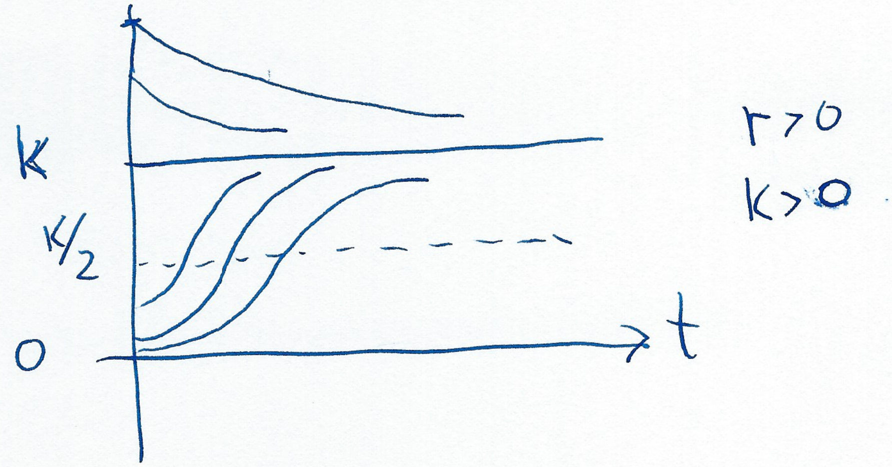
$y(0) = y_0$ is initial population

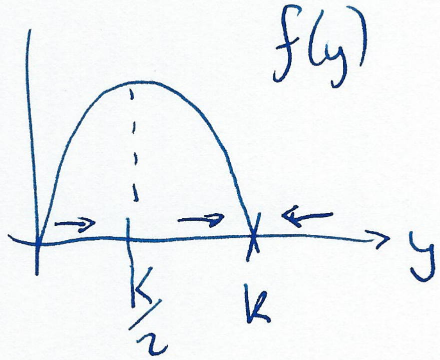
$$y_0 = \frac{CK}{C+K} \quad \therefore \quad C = \frac{P_0 K}{K - P_0}$$

$$\therefore\ y = \frac{P_0 K}{K - P_0} \frac{K}{\frac{P_0 K}{K - P_0} + Ke^{-rt}} = \frac{P_0 K}{K - P_0} \frac{1}{\frac{P_0}{K - P_0} + e^{-rt}}$$

$$y = \frac{P_0 K}{K - P_0} \frac{1}{\frac{P_0 + (K - P_0)e^{-rt}}{K - P_0}} = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}}$$

$$y = \frac{P_0}{\frac{P_0}{K} + (1 - P_0/k)e^{-rt}}$$





2nd Order Rate Equations in Chemical Kinetics

A typical chemical reaction equation might look like

(†) $CH_3Cl + NaOH \rightarrow CH_3OH + NaCl$

which represents a reaction of methyl chloride and sodium hydroxide, leading to methyl alcohol and sodium chloride

(†) is 1 equation in 4 unknowns.

$A \equiv CH_3Cl$ $\quad B \equiv NaOH$

$C \equiv CH_3OH$ and $D = NaCl$.

But if we know A, B, C, we can find D.

Let $X(t) \equiv$ amount of C, formed

from $Y(t) \equiv$ amount of A

$Z(t) \equiv$ amount of B

$\Rightarrow$ amount not converted into C is $Y - X$ and $Z - X$

$\therefore$ the rate of formation of X:

$$\frac{dX}{dt} = k(Y-X)(Z-X)$$

k is a constant of proportionality in units of 1/time.

ex) this is example 2.2 from Zill:

A and B form C:

(#) * for each unit of A, 4 are needed of B

(#) * 30 units of C form in 10 minutes.

* The rate of reaction of C

(☆) $\frac{dX}{dt} \propto$ amounts of A and B remaining at time t.

* Initially there are 50 units of A
32 units of B

How much of C is present in 15 minutes?

I.C. $X(0) = 0$ $\quad X(10) = 30$.

if $\underbrace{a}_{\text{amount of A}} + \underbrace{b}_{\text{amount of B}} = c = \text{amount of C}$

and $b = 4a$ (see †)

Hence $a + 4a = c \Rightarrow a = \frac{c}{5}$ & $b = \frac{4c}{5}$

$\therefore$ so $\frac{X}{5}$ units of A & $X\frac{4}{5}$ units of B

At time t the amounts remaining

are $\underbrace{50 - \frac{X}{5}}_{\text{for A}}$ $\quad \underbrace{32 - \frac{4X}{5}}_{\text{for B}}$

By (✩) $\displaystyle \frac{dX}{dt} = k\left(50 - \frac{X}{5}\right)\left(32 - \frac{4X}{5}\right)$

factor $\frac{1}{5}$ in RHS

$$\frac{dX}{dt} = \frac{k}{5}(250-X)\left(32-\frac{4X}{5}\right)$$

factor $\frac{4}{5}$ in RHS

$$\frac{dX}{dt} = k(250-X)(40-X)$$

or

$$\frac{dX}{(250-X)(40-X)} = k\,dt$$, separable.

Using partial fractions:

$$\frac{1}{(250-X)(40-X)} = \frac{A}{250-X} + \frac{B}{40-X}$$

$$1 = A(40-X) + B(250-X)$$

$$1 = A40 + B250 - X(A+B)$$

or $-(A+B) = 0$ & $40A + 250B = 1$

$$\therefore \quad -\int \frac{1/210}{250-X}\,dx + \int \frac{1/210}{40-x}\,dx = \int k\,dt$$

$$\ln\left|\frac{250-X}{40-X}\right| = 210kt + C$$

exponentiating:

$$\frac{250-X}{40-x} = \mathcal{k} e^{210kt}$$

$k, \mathcal{k}$ are still unknown:

Use the I.C. $X(0) = 0$

$$\frac{250}{40} = \mathcal{k}$$

$$\frac{250-X}{40-X} = \frac{25}{4} e^{210kt}$$

We then use (*) at $t=10$ $X=30$

$$\frac{\frac{25}{4} - 3}{4 - 3} = \frac{25}{4} e^{210k10}$$

$$22 = \frac{25}{4} e^{2100k}$$

$$\frac{88}{25} = e^{2100k}$$

logs of both sides:

$$\log\left(\frac{88}{25}\right) = 2100k$$

$$k = -0.125$$

$$\therefore \quad X(t) = 1000 \, \frac{1 - e^{-0.125t}}{25 - 4e^{-0.125t}}$$

$$\lim_{t \to \infty} X(t) = \frac{1000}{25} = 40$$

So 40 units of C are formed //