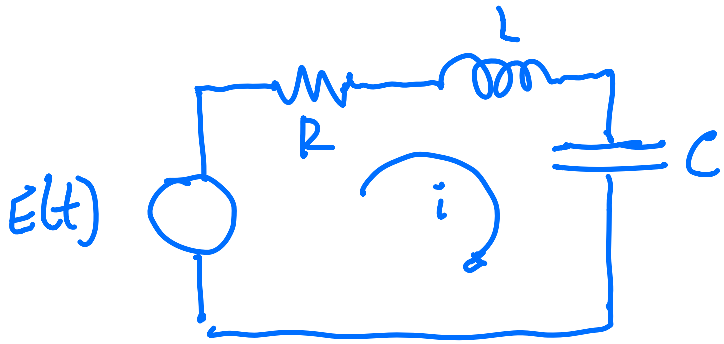
3.2 THE LRC CIRCUIT



Current
$i = \frac{dq}{dt}$ [Amperes]

q is charge [Coulombs]

t is time.

[Volts] $E(t)$ electromotive force
(Voltage) Potential (can be time dependent).

CONSERVATION OF ENERGY:

$$(\ast) \quad E(t) = Ri + L\frac{di}{dt} + \frac{q}{C}$$, $E(t)$ is given

constants
- R is resistor [Ω ohms]
- L is inductor [H Henry]
- C is capacitor [F Farads]

We can also write $(\ast)$ as

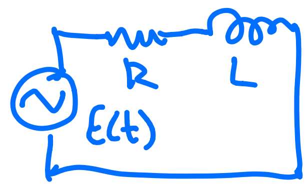
$$(\ast\ast) \quad E(t) = R\frac{dq}{dt} + L\frac{d^2}{dt^2}q + \frac{q}{C}$$

this is a 2nd order ODE for $q(t)$

If LRC only contains LR, then ($\cancel{C}$):

Find current, for $t \geq 0$

$R = 200\Omega \quad L = 10\,H$



$E(t) = 100 \sin t \quad t \geq 0$

Using conservation of energy (Kirchhoff Law)

$$E = Ri + L\frac{di}{dt} \quad \text{or:}$$

(ODE) $$100 \sin t = 200 i + 10 \frac{di}{dt}$$

$$\frac{di}{dt} + 20 i = 10 \sin t$$ (a linear 1st order ODE)

the (ODE) plus initial condition on $i(t)$ leads to a analytic unique solution.

We can also find $q(t)$ via $\frac{dq(t)}{dt} = i(t)$ (integrate) //

Newton's Law of Cooling

$$\frac{dT}{dt} \propto T - T_0$$

T is temprature $T(t)$
t is time dependent
T_0 is a constant (ambient/reference temperature)

$$\frac{[\text{Temp}]}{[\text{time}]} = \frac{dT}{dt} \propto T - T_0 \quad [\text{temp}]$$

$$(\ddagger) \qquad \frac{dT}{dt} = k(T - T_0) \qquad k = \left[\frac{1}{\text{time}}\right]$$

$k \leq 0$; $t \geq 0$

k is a constant.

T_0 is also called the steady state temperature (It is constant)

$$\lim_{t \to \infty} T(t) = T_0$$

"steady state temperature is the ambient temperature"

Solving $(\ddagger)$:

$$(\exists) \qquad \frac{dT}{(T - T_0)} = k\,dt \qquad , t > 0$$

is separable, so we know how to solve.

You need $T(t_0)$ an initial condition to find a unique solution.

Solving $(\circledast)$

$$\ln|T-T_0| = kt + c$$

$$T - T_0 = Ke^{kt}$$

$$(\divideontimes) \qquad T(t) = Ke^{kt} + T_0$$

$$\lim_{t\to\infty} T(t) = \lim (Ke^{kt} + T_0)$$

$$= T_0 \text{ because } k<0.$$

ex) Pull out pie from oven set at 350°F. Room temperature = 70°F. After 1 hour pie temperature is 175°F. The pie can be consumed when temperature is 110°F. How long do we have to wait?

Let $T(t)$ be temperature at $t \geq 0$

$[T] = [°F]$, $t = [hr]$

$T(0) = 350$

$T_0 = 70$

$T(1) = 175$

We can find k by knowing that (†) and I.C. has a unique solution

$$\frac{dT}{dt} = k(T-70)$$

$$T(t) = Ke^{kt} + 70$$

We have 2 unknowns, also have 2 constraints: $T(0)$, $T(1)$.

$T(0) = 350 = K + 70 \Rightarrow K = 280°F$

Use $T(1) = 175 = 280e^{k1} + 70$

$$105 = 280e^{k}$$

$$\ln\left(\frac{105}{280}\right) = k = -0.98\ldots$$

$$T(t) = 280e^{-0.98t} + 70 \quad , t \geq 0$$

To find t_e: $110-70 = 40 = 280\, e^{-0.98 t_e}$

or $\frac{1}{7} = e^{-0.98 t_e}$ $\therefore$ $t_e = \frac{-1}{0.98} \ln\left(\frac{1}{7}\right) = 1.91$ hrs.

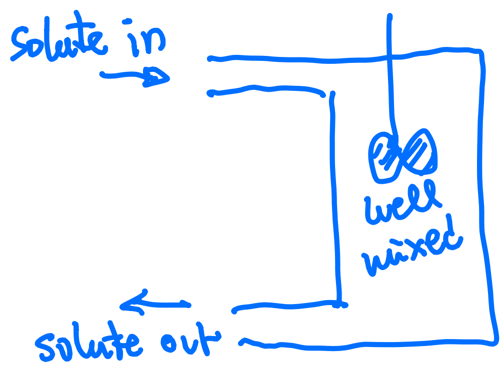
RADIATION (DECAY) $y(t)$ is amount of radiation at time t. The observation is that

$$\begin{cases} \dfrac{1}{y}\dfrac{dy}{dt} = \alpha, \quad \alpha < 0, \quad t > 0 \quad \text{(separable ODE)} \\ y(t_0) = y_0 \end{cases}$$

α is constant in units of 1/time.

yields a unique solution

Concentration of chemical species in tanks (and lakes, pools, etc).



A tank holds 500 gal. It has 100 gal brine ($H_2O + NaCl$).

Brine containing 1 lb salt/gal is pumped into tank at a rate of 3 gal/min.

The well mixed solution is drained at 2 gal/min.

Find $A(t)$ the salt [lb] as a function of time:

$\left[\frac{lb}{min}\right]$ $\frac{dA}{dt}$ = rate in - rate out

Rate in: $\frac{1\,lb}{gal} \cdot \frac{3\,gal}{min} = 3\,lb/min$

Rate out: accumulation is constant in time $(3-2)$ gal/min = 1 gal/min

$\left[\frac{lb}{min}\right]$ of $A(t)$]

$$\frac{2\,gal}{min}\left[\frac{A(t)}{100+t}\right]\frac{lb}{gal}$$

$$\Rightarrow \frac{dA}{dt} = 3 - \frac{2A}{100+t}$$

linear 1st order ODE:

$$\frac{dA}{dt} + \frac{2A}{100+t} = 3$$

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