Exact First Order ODE

Consider

$$\psi(x,y) = c \quad (\exists)$$

where c is a constant.

$\psi(x,y)$ is continuous and differentiable in x and y.

Differentiate $(\exists)$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad (\$)$$

$d\psi$ is an **exact differential**

A special case of ($) is when

$$\frac{\partial \psi}{\partial x} = M(x) \qquad \frac{\partial \psi}{\partial y} = N(y)$$

$$d\psi = M(x)\,dx + N(y)\,dy = 0$$

then $d\psi$ is a separable ODE.

More generally, however,

$$\text{A} \quad \frac{\partial \psi}{\partial x} = M(x,y) \qquad \frac{\partial \psi}{\partial y} = N(x,y) \quad \text{B}$$

$$(\#)\quad d\psi = M(x,y)\,dx + N(x,y)\,dy = 0.$$

Differentiate A wrt y

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial}{\partial y} M(x,y) \qquad \text{C}$$

Differentiate B wrt x:

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial}{\partial x} N(x,y) \quad D$$

So C & D are the same if

$$d\psi = M(x,y)\,dx + N(x,y)\,dy = 0$$

$$= \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0$$

So for a ODE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

to be an exact differential

Then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (★)

if so then

$d\psi = M\,dx + N\,dy$

and integrating yields $\psi = C$ //

ex) $\begin{cases} 2x - 5y + (3y^2 - 5x)\dfrac{dy}{dx} = 0 & \text{ODE} \\ y(0) = 1 & \text{I.C.} \end{cases}$

put into the form (#)

$$(\#) \quad \underbrace{(2x - 5y)}_{M(x,y)}dx + \underbrace{(3y^2 - 5x)}_{N(x,y)}dy = 0$$

To see whether (#) is $d\psi = 0$

the test: Use (#)

See if $\dfrac{\partial M}{\partial y} \stackrel{?}{=} \dfrac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = -5 \qquad \frac{\partial N}{\partial x} = -5$$

Conclude that (#) is an exact differential:

$d\psi = 0$, so integrate to find $\psi = c$.

$$\underbrace{(2x-5y)}_{M(x,y)}dx + \underbrace{(3y^2-5x)}_{N(x,y)}dy = 0$$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

So $\frac{\partial \psi}{\partial x} = M(x,y) = 2x-5y \quad (*)$

$\frac{\partial \psi}{\partial y} = N(x,y) = 3y^2-5x \quad (**)$

Integrate $(*)$ wrt x:

$(***)\quad \psi = x^2 - 5xy + h(y)$

To find $h(y)$, using $(**)$, diff $(***)$ wrt y:

$$\frac{\partial \psi}{\partial y} = -5x + \frac{dh}{dy} = -5x + 3y^2$$

by $(**)$

$\Rightarrow \frac{dh}{dy} = 3y^2$ integrate wrt y:

(****) $h(y) = y^3 + c$

Putting (***) & (****):

$$\psi = x^2 - 5xy + y^3 = C$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

Apply I.C.

$$y(0) = 1$$

$$1 = C$$

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Integrating Factors

It's sometimes possible to convert a 1st order ODE that is not exact to an exact ODE. Find an integrating factor:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

(this may/may not be exact)

if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (not exact)

Find $\mu(x,y)$ the integrating factor

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

s.t.

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

so $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$

(*) $\frac{\partial}{\partial x}(\mu M)dx + \frac{\partial}{\partial y}(\mu N)dy = 0$

$\Rightarrow \psi = C$

So $Mdx + Ndy = 0$ may not

be exact, but $\mu M dx + \mu N dy = 0$ is. Hence

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

where $\frac{\partial \psi}{\partial x} = \mu M$ and $\frac{\partial \psi}{\partial y} = \mu N$

so long as

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}(\mu M) = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}(\mu N)$$

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y} M - \frac{\partial \mu}{\partial x} N = \mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \quad (\star)$$

OK, so μ must satisfy $(\star)$ but how do we find μ?

At this point, we need to make guesses.

How about proposing that $\mu = \mu(x)$? (whether this is a good guess or not, you need to follow it through to see if it works: that you can find such an $\mu(x)$:

So, if $\mu = \mu(x)$

$$-\frac{\partial \mu}{\partial x} N = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

or $$\frac{d\mu}{\mu} = -\frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\ln \mu = -\int \frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx$$

Alternatively, if $\mu = \mu(y)$

$$\frac{d\mu}{dy} = \frac{\mu}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy \, .$$

ex) $$\frac{dy}{dx} = e^{2x} + y + 1$$

$$dy - (e^{2x} + y - 1)dx = 0 \qquad (\$)$$

$N = 1 \qquad M = -(e^{2x} + y + 1)$

Test for exactness

$$\frac{\partial N}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = -1 \quad \Big\}\ \text{Not exact: } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Let us try $\mu = \mu(x)$

(i) Multiply ($) by $\mu(x)$

$$\mu(x)\,dy - \mu(x)\left(e^{2x}+y-1\right)dx = 0$$

$$\frac{\partial \psi}{\partial y}\,dy + \frac{\partial}{\partial x}\psi\,dx = 0$$

(A) $\dfrac{\partial \psi}{\partial y} = \mu(x)$; $\dfrac{\partial \psi}{\partial x} = -\mu(x)\left(e^{2x}+y-1\right)$ (B)

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{d\mu}{dx} \ ; \ \frac{\partial^2 \psi}{\partial y \partial x} = -\mu(x)\frac{\partial}{\partial y}\left(e^{2x}+y-1\right)$$

$$\therefore \frac{d\mu}{dx} = -\mu(x)\frac{\partial}{\partial y}\left(e^{2x}+y-1\right) = -\mu(x)$$

$$\frac{d\mu}{dx} = -\mu(x) \quad \text{or} \quad \frac{d\mu}{\mu} = -dx$$

$$\ln|\mu| = -x + c \Rightarrow \mu = e^{-x} \quad (\#)$$

(we dispense with the constant)

$\therefore$ Plug $(\#)$ into (A):

$$\frac{\partial \psi}{\partial y} = e^{-x} \quad \text{integrating in } y:$$

$$(C) \quad \psi = ye^{-x} + h(x) \qquad \text{Differentiate C wrt } x:$$

$$\frac{\partial \psi}{\partial x} = -e^{-x}y + h'(x) = -e^{-x}(e^{2x} + y - 1) \text{ using } (B).$$

$$\cancel{-e^{-x}y} + h'(x) = -e^{x} \cancel{-ye^{-x}} + e^{-x}$$

$$\therefore \quad h'(x) = -(e^{x} - e^{-x}) = -2\sinh x$$

integrating $h'(x)$ wrt x:

$$h(x) = -2\cosh x + c$$

Plugging $h(x)$ in (C):

$$\psi = ye^{-x} - 2\cosh x + c$$

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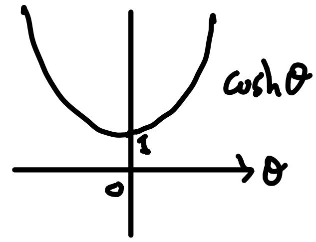
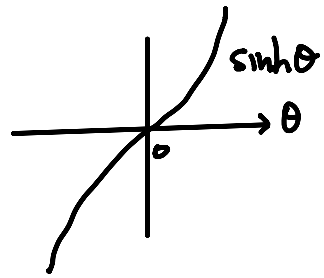
Hyperbolic Functions

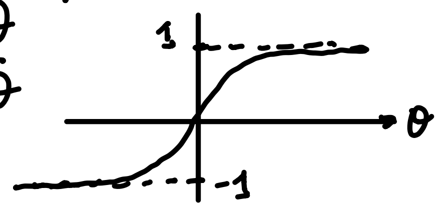
$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$
(even)



$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$
(odd)



$$\tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} = \frac{\sinh\theta}{\cosh\theta}$$



$$\frac{d}{d\theta}\cosh\theta = \sinh\theta$$

$$\frac{d}{d\theta}\sinh\theta = \cosh\theta$$

TRIGONOMETRIC FORMULAS:

Euler's Formulas $\quad i = \sqrt{-1}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

e.g. $\frac{d}{d\theta}\cos\theta = \frac{1}{2}\frac{d}{d\theta}e^{i\theta} + \frac{1}{2}\frac{d}{d\theta}e^{-i\theta} = \frac{i}{2}\left(e^{i\theta}-e^{-i\theta}\right) = \frac{-1}{2i}\left(e^{i\theta}-e^{-i\theta}\right) = -\sin\theta$ //

Another example, exact equation:

$$x \frac{dy}{dx} = 2xe^{x} - y + 6x^2$$

① Show it is exact:

if $N\,dy + M\,dx = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

② Show that $\psi = xy + h(x)$

③ show that $h'(x) = -2xe^{x} - 6x^2$

④ show that $\psi = xy - 2xe^{x} - 2e^{x} - 2x^3 + C$ //